

- Set up the problem w/ $P()$ in terms of X .

- Watch out! for discrete + continuous setups!

Eg: $X \sim f(x) = |x| - 1 \leq x \leq 1$
if $Y = X^2$, find the pdf of Y , $f(y)$

Set up def of CDF for continuous

$$F(y) = P(Y \leq y) = P(X^2 \leq y)$$

Now express as X

$$P(X^2 \leq y) = P(-\sqrt{y} \leq X \leq \sqrt{y})$$

$$= \int_{-\sqrt{y}}^{\sqrt{y}} |x| dx = y \quad (\because \text{cont})$$

BUT!! This is disc + cont!

$$\therefore F(y) = \begin{cases} 0 & \text{if } y \leq 0 \\ 1 & \text{if } y \geq 1 \\ y & \text{if } 0 < y < 1 \end{cases}$$

to get pdf, ~~$f(y) = \frac{d}{dy} F(y) = 1$~~

differentiate, $f(y) = F'(y) = 1$

in this mixed case,

$$f(y) = \begin{cases} 0 & \text{if } y \leq 0 \text{ or } y \geq 1 \\ 1 & \text{if } 0 < y < 1 \end{cases}$$

BIVARIATE CASE

There is discrete and cont. of course

DISC

$$0 \leq f(x, y) \leq 1$$

$$\sum_x \sum_y f(x) f(y) = 1$$

$$A \subseteq \mathbb{R}^2 \Rightarrow P(X, Y \in A) = \sum_{(x, y) \in A} f(x) f(y)$$

CONT

$$f(x, y) \geq 0$$

$$\iint_{\mathbb{R}^2} f(x) f(y) = 1$$

$$P(A) = P(X, Y \in A) = \iint_A f(x, y) dy dx$$

This is like

$$f(x, y) dy dx = P(x < X < x+dx, y < Y < y+dy)$$

Unit circle $C \equiv (x - \frac{1}{2})^2 + (y - \frac{1}{2})^2 \leq \frac{1}{4}$

$$\boxed{\text{Circle}} \quad \text{Area} = \pi/4$$

$$\therefore \text{pdf} = f(x, y) = \begin{cases} \frac{4}{\pi} & \text{if } (x, y) \in C \\ 0 & \text{otherwise} \end{cases}$$

$$\therefore \text{dart} \rightarrow \frac{\# \text{ in circle}}{\# \text{ in sq}} = \frac{\frac{4}{\pi} \cdot \pi/4}{1} = \frac{1}{4}$$

$$\therefore \pi = 4 \cdot \frac{\# \text{ in circle}}{\# \text{ in square}}$$

w/ Monte Carlo!

$$\text{Area of circle} = \pi/4$$

$$\text{Area of sq} = 1$$

??? Limit ???

①

(1)

Joint and Marginal PDF's and CDF's
(This is not too bad and needs practice)

CDF $\xrightarrow{\text{partial derivatin}}$ PDF
 $\xleftarrow{\text{Integrate wrt to } x \text{ or } y}$

CDF $F(x, y)$

- It is non-decreasing!
- $\lim_{x \rightarrow -\infty} F(x, y) = \lim_{y \rightarrow -\infty} F(x, y) = 0$
- $P(Y \leq y) = \lim_{x \rightarrow +\infty} F(x, y) = F_Y(y)$
- Similarly, $P(X \leq x) = \lim_{y \rightarrow +\infty} F(x, y) = F_X(x)$
- and, $\lim_{x \rightarrow +\infty} \lim_{y \rightarrow +\infty} F(x, y) = 1$

Eg: CDF $F(x, y) = \begin{cases} 1 - e^{-x} - e^{-y} + e^{-(x+y)} & \forall x \geq 0, y \geq 0 \\ 0 & \text{otherwise} \end{cases}$

Marg. CDF of X

$$F_X(x) = \lim_{y \rightarrow \infty} F(x, y) = \begin{cases} 1 - e^{-x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

pdf (joint), $f(x, y) = \frac{\partial^2}{\partial x \partial y} [F(x, y)] = e^{-(x+y)}$
 $\forall x \geq 0, y \geq 0$

Marginal

"Everything but the variable you want"

if $f(x, y)$ is DISCRETE,

$$\left. \begin{aligned} f_X(x) &= \sum_y f(x, y) \\ f_Y(y) &= \sum_x f(x, y) \end{aligned} \right\} \text{Marginal pmf}$$

if CONTINUOUS,

$$\left. \begin{aligned} f_X(x) &= \int_{\mathbb{R}} f(x, y) dy \\ f_Y(y) &= \int_{\mathbb{R}} f(x, y) dx \end{aligned} \right\} \text{Marginal pdf}$$

$$\begin{aligned} e^x &= \infty \text{ if } x \rightarrow +\infty \\ e^x &= 0 \text{ if } x \rightarrow -\infty \end{aligned}$$

→ FUCKING LIMITS ARE IMPORTANT

Eg: $f(x, y) = \begin{cases} \frac{2}{4} x^2 y & \text{if } x^2 \leq y \leq 1 \\ 0 & \text{otherwise} \end{cases}$
JOINT PDF

So! $f_X(x) = \int_{\text{limits of } y!} \frac{2}{4} x^2 y \cdot dy$
MARGINAL PDF
 $= \int_{x^2}^1 \frac{2}{4} x^2 y \cdot dy = \frac{2}{4} x^2 (1 - x^4) \quad \forall -1 \leq x \leq 1$

and $f_Y(y) = \int_{-\sqrt{y}}^{\sqrt{y}} \frac{2}{4} x^2 y \cdot dx$
 $x^2 = y \Rightarrow x = \pm \sqrt{y}$
 $= \frac{7}{2} y^{5/2} \quad 0 \leq y \leq 1$

(???) Limits ?

CONDITIONAL

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

So, $P(Y=f(y) | X) = \frac{P(X=x \cap Y=f(y))}{P(X)}$

defined that: $P(X=x) = f_X(x)$

~~$P(A|B)$~~ So, $f(y|x) = \frac{f(x, y)}{f_X(x)}$

SEE EXAMPLE for discrete
continuous in hand

$$f(x, y) = \begin{cases} \frac{2}{4} x^2 y & , x^2 \leq y \leq 1 \end{cases}$$

$$f_X(x) = \frac{2}{4} x^2 (1 - x^4) \quad \forall -1 \leq x \leq 1$$

$$f_Y(y) = \frac{7}{2} y^{5/2} \quad \forall 0 \leq y \leq 1$$

$$\begin{aligned} f(y|x) &= \frac{f(x, y)}{f_X(x)} = \frac{\frac{2}{4} x^2 y}{\frac{2}{4} x^2 (1 - x^4)} \\ &= \frac{2y}{1 - x^4} \quad , x^2 \leq y \leq 1 \end{aligned}$$

- ⑤ Problems are like "Given $f(y|x)$, $f_x(x)$, find $f_Y(y)$ "
- ① find $f(x, y) = f(y|x) \cdot f_x(x)$
 - ② find $f_Y(y) = \int_{\mathbb{R}} f(x, y) dx$

EG: $f_x(x) = 2x \quad 0 < x \leq 1$

$f(y|x) \sim \text{Unif}(0, x)$

find $f_Y(y)$



Uniform, so $f(y|x) = \frac{1}{x}, 0 < y < x$

$\therefore f(x, y) = f(y|x) \cdot f_x(x)$
 $= \left(\frac{1}{x}\right)(2x) = 2$
 WITH!!

now, $f_Y(y) = \int_y^1 2 dx = 2(1-y)$
 $0 < y \leq 1$

Independence

??? Discrete case:

Indep. iff $f(x, y) = f_x(x) f_Y(y)$

② iff for some $a(x)$ and $b(y)$,

$f(x, y) = a(x) b(y)$

→ LIMITS ARE A PROBLEM!!

Eg: if $f(x, y) = \begin{cases} 1 & \text{for } x \neq y \\ 0 & \text{if } x = y \end{cases}$

not independent!!

→ Eg $f(x, y) = \frac{1}{x+y}$

cannot decompose to $a(x)$
 not indep! $b(y)$

Independent ... so what?

SIDENOTE: whether or not indep,

$$E[X+Y] = E[X] + E[Y]$$

Generally (by induction),

$$E\left[\sum_{i=1}^n X_i\right] = \sum_{i=1}^n E[X_i]$$

But if independent,

$$E[XY] = E[X]E[Y]$$

$$\text{Var}(X+Y) = \text{Var}(X) + \text{Var}(Y)$$

→ generally,

$$\text{Var}\left[\sum X_i\right] = \sum \text{Var}(X_i)$$

$$\text{Var}\left[\sum a_i X_i + b_i\right] = \sum a_i^2 \text{Var}(X_i)$$

Random Samples

X_i is a set of random samples iff

- X_i 's are independent
- Each X_i has same pdf/pmf! $f(x)$
- ie, $X_i \stackrel{iid}{\sim} f(x) \quad i \in (1, n)$

$\therefore$ of independence, $E[\bar{x}] = \mu$ (proof)

$$\bar{x} \equiv \sum \frac{x_i}{n}, E[x_i] = \mu, \text{Var}[x_i] = \sigma^2$$

$$\text{and } \text{Var}[\bar{x}] = \frac{\sigma^2}{n} \text{ (proof...)}$$

note: Variance decreases!

⇒ $\bar{x}$ is a great estimator of μ ...

Cond Expectation

$$E[Y|X=x] \equiv \begin{cases} \sum_y y f(y|x) & \text{disc} \\ \int y f(y|x) dy & \text{cont} \end{cases}$$

$\Rightarrow$ function of x !

$\Rightarrow$ SEE EXAMPLE! FROM NOTES!

$$f(y|x) = \frac{f(x,y)}{f_x(x)}$$

Eg $f(x,y) = \frac{21}{4} x^2 y \quad x^2 \leq y \leq 1$

find $E[Y|X]$,

$$E[Y|X] = \int y f(y|x) dy$$

"answer is function of x !"

need $f_x(x)$, we have $f(x,y)$

$$f_x(x) = \int_{x^2}^1 \frac{21}{4} x^2 y dy$$

$$= \frac{21}{8} x^2 (1-x^4)$$

$$\therefore f(y|x) = \frac{\frac{21}{4} x^2 y}{\frac{21}{8} x^2 (1-x^4)} = \frac{2y}{1-x^4}$$

$$\therefore E[Y|X] = \int_{x^2}^1 \frac{2y^2}{1-x^4} dy = \frac{2}{3} \frac{1-x^6}{1-x^4}$$

So, $E[Y|X=0.5] = (\text{Sub } x)$

$$= \underline{\underline{0.70}}$$

Weird Shit: $E[E(Y|X)] = E(Y)$

This is a bit magical but ^{proof} we use to solve problems.

Eg: w/ $f(x,y) = \frac{21}{4} x^2 y \quad x^2 \leq y \leq 1$
find $E[Y]$

Two ways $\xrightarrow{①} E[Y] = \int y f_Y(y) dy$

$\xrightarrow{②} \int E(Y|x) f_x(x) dx$
(fuck me...)

①: Compute marginal pdfs

$$f_Y(y) = \int_{-\sqrt{y}}^{\sqrt{y}} f(x,y) dx$$

"want this to 'dissolve'"

② fuck it.

COVARIANCE

Independence $\implies$ Covariance

Not necessarily!

$$\text{Cov}(X,Y) \equiv \sigma_{XY} \equiv E \left[\begin{matrix} (X - E[X]) \\ \times (Y - E[Y]) \end{matrix} \right]$$

$$\text{Cov}(X,X) = E[(X - E[X])^2] = \text{Var}(X)$$

$$\text{Cov}(X,Y) = E[XY] - E[X]E[Y]$$

(if independent, this is zero!)

$$\therefore E[XY] = E[X]E[Y]$$

BUT DOES NOT IMPLY INDEPENDENCE!

Eg: $X \sim \text{Unif}(-1,1), Y = X^2$ } dependent!

$$E[X] = \int_{-1}^1 x \frac{1}{2} dx = 0, E[XY] = E[X^3] = 0$$

and $\text{Cov}(X,Y) = 0$

LOTUS: let X be cont. w/pdf $f(x)$

$$E[X] = \int_{\mathbb{R}} f(x) \cdot x \, dx$$

if $Y = \text{some function } h(X)$,

LOTUS says $E[Y] = \int_{\mathbb{R}} h(x) f(x) \, dx$

no x !
replace w/func.

so, k^{th} moment is

$$E[X^k] \xrightarrow{\text{LOTUS}} \int_{\mathbb{R}} x^k f(x) \, dx$$

and k^{th} Central Moment is

$$E[(X-\mu)^k] \xrightarrow{\text{LOTUS}} \int_{\mathbb{R}} (x-\mu)^k f(x) \, dx$$

and $\text{Var}(X) \equiv 2^{\text{nd}}$ Central Moment $\heartsuit$

$$\text{Var}[X] \equiv E[(X-\mu)^2] = E(X^2) - [E(X)]^2$$

IN TIMES OF TROUBLE.... $h(x)$

Sometimes, this gets messy for weird $h(x)$

so use Taylor Series expansion!

if $\mu \equiv E[X]$, $\sigma^2 \equiv \text{Var}(X) \equiv \dots$ whatever

$$\text{then, } Y = h(X) \approx h(\mu) + (X-\mu)h'(\mu) + \frac{(X-\mu)^2 h''(\mu)}{2!} + R$$

LOTUS

$$\Rightarrow E[Y] = h(\mu) + E[X-\mu] h'(\mu) + E[(X-\mu)^2] \cdot \frac{h''(\mu)}{2!} + R$$

$$\text{ie, } E[Y] = h(\mu) + h''(\mu) \cdot \frac{\sigma^2}{2}$$

$E[X]$

and $\text{Var}[Y] = \text{Super crude, drop } 3^{\text{rd}} \text{ term}$
 $\text{Var}[h(\mu) + h'(\mu)(X-\mu)]$

$$\text{Var}[Y] \approx [h'(\mu)]^2 \sigma^2$$

Moment Gen Functions

They do some useful things...

First, define MGF

$$M_X(t) = E[e^{tx}]$$

MGF is a func. of t !! NOT of x !

Eg: $X \sim \text{Bern}(p)$ $X = \begin{cases} 0 & \text{w/prob } 1-p \\ 1 & \text{w/prob } p \end{cases}$

$$\Rightarrow \text{MGF} \equiv M_X(t) = E[e^{tx}] \quad \text{LOTUS!}$$

$$= \sum_x e^{tx} f(x) = e^{t(0)}(1-p) + e^{t(1)}(p)$$

$$\Rightarrow M_X(t) = e^{t(0)}q + e^{t(1)}p$$

$$\text{or } M_X(t) = pe^t + q (=1-p+pe^t)$$

what's the use?? under "some technical condition",

$$E[X^k] = \left. \frac{d^k}{dt^k} M_X(t) \right|_{t=0} \quad \forall k=1,2,\dots$$

"the k^{th} moment is the k^{th} derivative of the MGF"

Eg: $X \sim \text{Exp}(\lambda)$ and $M_X(t) = \frac{\lambda}{\lambda-t} \quad +\lambda>t$

$$\Rightarrow \mu = E[X] = \frac{d}{dt} \left(\frac{\lambda}{\lambda-t} \right) = \frac{\lambda}{(\lambda-t)^2}$$

with $t=0$, $\mu = \frac{\lambda}{\lambda^2} = \frac{1}{\lambda}$

||| $E[X^2] = \frac{d^2}{dt^2} \left(\frac{\lambda}{\lambda-t} \right) = \frac{2\lambda}{(\lambda-t)^3}$

with $t=0$, $E[X^2] = \frac{2}{\lambda^2}$

$$\Rightarrow \text{Var}[X] = E[X^2] - \mu^2 = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}$$

Now: if X has mgf $M_X(t)$

and Y is linear $= aX + b$

what is mgf of Y , $M_Y(t)$?

$$\rightarrow M_Y(t) = E[e^{tY}] = E[e^{atX + bt}]$$

$$= E[e^{atX}] \cdot E[e^{bt}] \quad \text{constant}$$

$$= e^{bt} \cdot E[e^{(at)X}]$$

$$= e^{bt} \cdot M_X(at)$$

IMPORTANT!

Sample app: $X \sim \text{Exp}(\lambda)$

what is mgf of $Y = 3X + 2$?

$$M_Y(t) = e^{2t} M_X(3t)$$

$$\text{mgf of } X = \frac{\lambda}{\lambda - t} = M_X(t)$$

$$\Rightarrow M_X(at) = M_X(3t) = \frac{\lambda}{\lambda - 3t}$$

$$\text{So! } M_Y(t) = e^{2t} \cdot \frac{\lambda}{\lambda - 3t}$$

MGF's are a "fingerprint"
to identify distributions

GO REVERSE: if $M_Y(t) = e^{2t} \frac{\lambda}{\lambda - 3t}$

then you know things about
 X and its function Y

$\sim \text{Exp}(\lambda)$

$(3X + 2)$

RV Problems: You have X
You know pdf/pmf

$Y = h(X)$ — what's pdf/pmf?

$\Rightarrow$ LOTUS is only for $E[Y]$,
not the entire dist!

no! we get the pdf/pmf of Y not $E[Y]$

① if X is discrete $\rightarrow Y_{\text{discrete}}$
 $h(X)$

formally,

$$\text{pmf of } Y = g(y) = P(Y=y) = P(h(X)=y)$$

$$= P(\{x | h(x)=y\}) = \sum_{x | h(x)=y} f(x)$$

WTF?

This is step 2 — simple w/ a table!

② if X is cont $\rightarrow Y$ is discrete
or Y is cont.

here, compute the CDF — ①

then differentiate — ②

that's it. CDF $G(Y) = P(Y \leq y)$

$$= P(h(X) \leq y) = \int_{\{x | h(x) \leq y\}} f(x) dx$$

$$\text{then, pdf of } Y = g(Y) = \frac{d}{dy} G(Y)$$

YOU NEED EXAMPLES!

(N)

THE NORMAL DIST ♥

AKA
"Gaussian"

$X \sim \text{Nor}(\mu, \sigma^2)$ iff it's pdf is

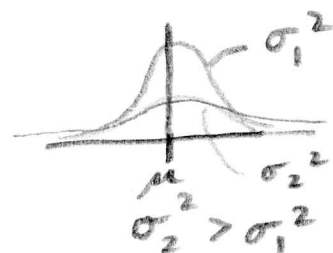
$$\rightarrow f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{\left[-(x-\mu)^2/2\sigma^2\right]} \quad \forall x \in \mathbb{R}$$

$$\rightarrow F(x) = \int_{-\infty}^x (\quad) = \wedge \quad \text{no closed-form solution....}$$

$$\rightarrow \int_{\mathbb{R}} f(x) dx = 1 \quad (\heartsuit\heartsuit\heartsuit)$$

$$\rightarrow E[X] = \mu \quad \text{Var}[X] = \sigma^2$$

$$\rightarrow \text{mgf} = M_X(t) = \exp\left(\mu t + \frac{1}{2}\sigma^2 t^2\right)$$



$\rightarrow$ Combo-time! If $X_1, X_2, X_3, \dots, X_n$

are independent and $N(\mu_i, \sigma_i^2) \quad i \in (0, n)$

then! $Y \equiv \sum_{i=1}^n a_i X_i + b_i \sim N\left(\sum_{i=1}^n a_i \mu_i + b_i, \sum_{i=1}^n a_i^2 \sigma_i^2\right)$ (proof by mgf...)

ADDITIVE PROPERTY!!

$$\text{Eg: } X \sim N(3, 4) \quad \text{and} \quad Y \sim N(4, 6) \quad \left\{ \begin{array}{l} E[2X-3Y] = 2(3) - 3(4) = -1 \\ \text{Var}[2X-3Y] = 2^2(4) + 3^2(6) = 70 \end{array} \right.$$

$\Rightarrow 2X-3Y$ is also $\sim N(-1, 70)$

VARIANCES ALWAYS ADD
(constant UP!)

$$X \sim N(\mu, \sigma^2) \Rightarrow aX+b \sim N(a\mu+b, a^2\sigma^2)$$

2. Why? new mean = $E[aX+b] = aE[X] + E[b] = a\mu + b$
new var = $\text{Var}[aX+b] = a^2\sigma^2$

So define RV, $Z \equiv \frac{X-\mu}{\sigma}$ THIS IS Norm(0,1) !!!

The Standard Normal Dist! ♥♥♥

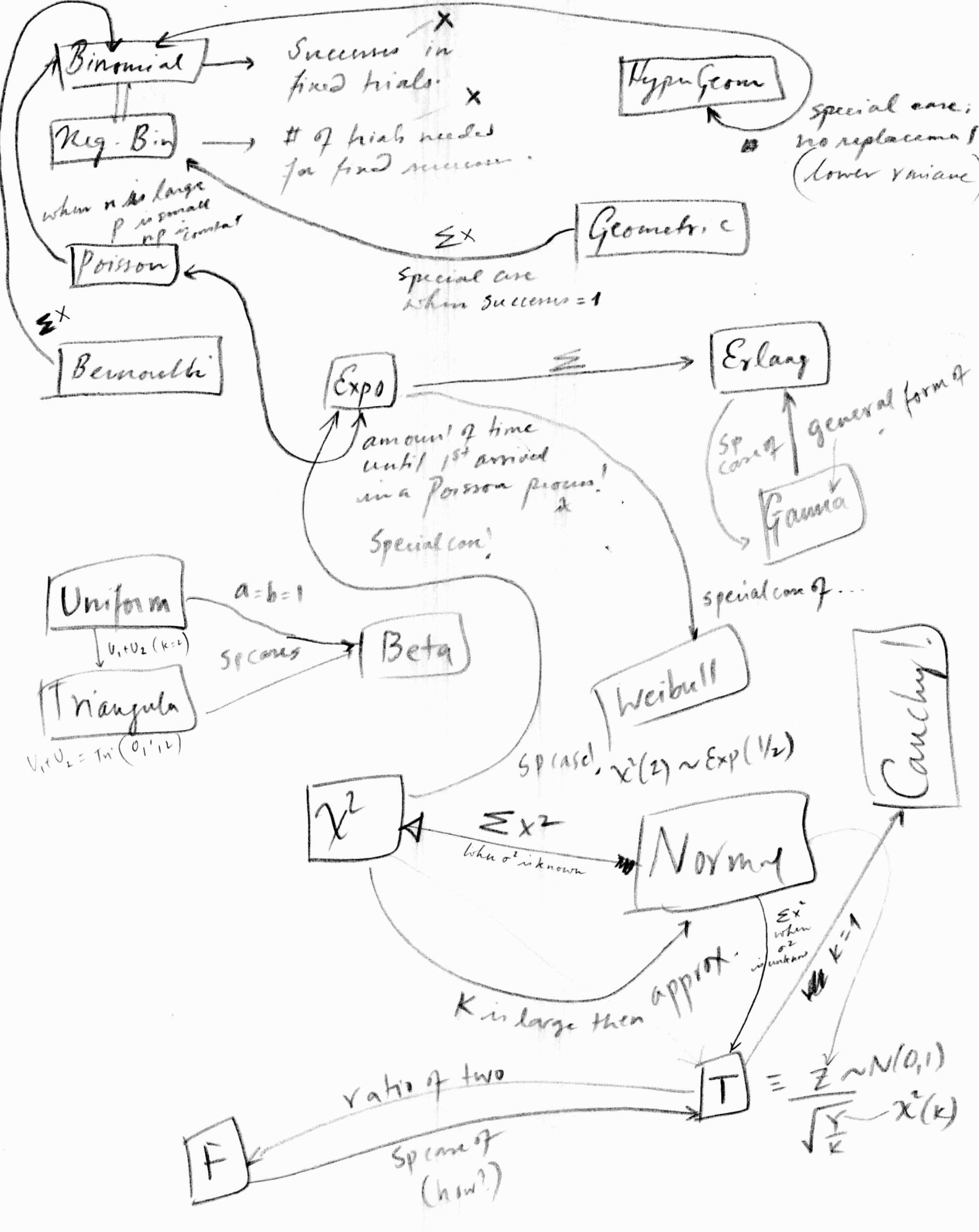
<u>List</u>	pdf / pmf CDF	$E[X]$	$Var[X]$	Mgf
<u>GAMMA</u> $\Gamma(\alpha) = \int_0^{\infty} t^{\alpha-1} e^{-t} dt$ PDF $\Gamma(1/2) = \sqrt{\pi}$ (lol...) if α is a +ve integer $\Gamma(\alpha) = (\alpha-1)!$	$f(x) = \frac{\lambda^\alpha e^{-\lambda x} x^{\alpha-1}}{\Gamma(\alpha)} \quad \forall x > 0$	$\frac{\alpha}{\lambda}$	$\frac{\alpha}{\lambda^2}$	
<u>TRIANGULAR</u> $T(a, b, c)$ use to model RV's based on limited data Eg: mean, mode, max	$f(x) = \begin{cases} \frac{2(x-a)}{(b-a)(c-a)} & a < x \leq b \\ \frac{2(c-x)}{(c-b)(c-a)} & b < x < c \\ 0 & \text{otherwise} \end{cases}$	$\frac{a+b+c}{3}$		LOLZILLA (it's a fucking mess!)
<u>BETA</u> use to model RV's restricted to interval (a, b) Beta function $\beta(a, b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$ $= \int_0^1 x^{a-1} (1-x)^{b-1} dx$	$f(x) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} x^{a-1} (1-x)^{b-1}$ $0 < x < 1$	$\frac{a}{a+b}$	$\frac{ab}{(a+b)^2(a+b+1)}$	
	Very flexible! Can get Exp, Unif, Normal out of this....			
<u>WEIBULL</u> scale a shape b use for reliability models	$f(x) = ab(ax)^{b-1} e^{-(ax)^b}$ $\forall x > 0$ $F(x) = 1 - \exp[-(ax)^b]$ $\forall x > 0$	$\left(\frac{1}{a}\right) \Gamma\left(1 + \frac{1}{b}\right)$		LOLZILLA a mess...
<u>CAUCHY!</u> use in Maths stat. to disprove thing!	$f(x) = \frac{1}{\pi(1+x^2)}$ $F(x) = \frac{1}{2} + \frac{\tan^{-1}(x)}{\pi}$ $x \in \mathbb{R}$	Undefined	∞	LOL
WEIRD: $X_i \stackrel{iid}{\sim} \text{Cauchy} \Rightarrow \frac{\sum X_i}{n} \sim \text{Cauchy!}$ <i>End back where you began.....</i>				

Dist	$f(x) = P(X=x)$	$E[X]^n$	$Var[X]^{\sigma^2}$	MGF $M_X(t)$
Poisson *	$\frac{\lambda^x e^{-\lambda}}{x!}$	λ	λ	$e^{\lambda(e^t - 1)}$
$P_n(t) \equiv P(N(t)=n)$ $P(X) = \sum_{x=0}^{\infty} P(X=x)$ $X_1, X_2, \dots, X_n$ ADDITIVE! if $Y = X_1 + X_2 + \dots + X_n$ then $Y \sim \text{Poisson}(\sum \lambda_i)$ Inter-arrival time is EXPONENTIAL!! ADJUST λ to match the question! $\lambda_1 = \frac{3}{hr}, \lambda_2 = \frac{5}{hr} \Rightarrow \lambda_{total} = \frac{4}{30 \text{ minutes}} (\frac{5}{hr})$				
independent! identically distributed! proof is mfg!				

UNIF	$f(x) = \begin{cases} \frac{1}{b-a} & a \leq x \leq b \\ 0 & \text{oth} \end{cases}$ $F(x) = \begin{cases} 0 & x < a \\ \frac{x-a}{b-a} & a \leq x \leq b \\ 1 & x > b \end{cases}$	$\frac{a+b}{2}$	$\frac{(a-b)^2}{12}$	$\frac{e^{bt} - e^{at}}{t(b-a)}$
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Exp(2)	$f(x) = \begin{cases} \lambda e^{-\lambda x} & x > 0 \\ 0 & \text{oth} \end{cases}$ $F(x) = 1 - e^{-\lambda x}$	$\frac{1}{\lambda}$	$\frac{1}{\lambda^2}$	$\frac{\lambda}{\lambda - t}, t < \lambda$
MEMORYLESS $P(X > s+t X > s) = P(X > t) = e^{-\lambda t}$ Eg: Bulb has avg life of 10 mo. Made it 20. Prob it will make it 8 more? $P(X > 28 X > 20) = P(X > 8) = e^{-\lambda(8)}$ $\lambda = \frac{1}{10 \text{ mo}}$ 1 every 10 months!				
Similar to GEOM (but continuous!) not a coincidence!				
SIDE NOTE: $P(X > t) = 1 - F(t)$ continuous 				

ERLANG *	$f(s) = \frac{\lambda^k e^{-\lambda s} s^{k-1}}{(k-1)!}, s \geq 0$ $Erlang_k(\lambda) \rightarrow Exp(\lambda)$ $S = \sum_{i=1}^k X_i$ just a sum of Expos	$\frac{k}{\lambda}$	$\frac{k}{\lambda^2}$	$\left(\frac{\lambda}{\lambda - t}\right)^k$
Eg: X, Y are iid $Exp(2)$ find $P(X+Y < 1)$ $\rightarrow$ cdf! $k=2, s=1$ $\Rightarrow P(X+Y < 1) = 1 - \sum_{i=0}^{2-1} \frac{e^{-2(1)} (2 \cdot 1)^i}{i!}$ $= 0.594$				



<u>P in Success!</u> <u>dist</u>	pdf pmf $f(x)$	cdf $F(x)$	E.V $E[X]$	Var $Var[X]$	MGF MGF $M_X(t)$
			p	pq	$pe^t + q$

<u>Bernoulli</u> Bern(p) *	$x = \begin{cases} 1 & \text{w.p. } p \\ 0 & \text{w.p. } q \end{cases}$ $f(x) = \begin{cases} p & \text{if } x=1 \\ q & \text{if } x=0 \end{cases}$				
<u>Binomial</u> "k success out of n trials" *	$P(Y=k)$ $= \binom{n}{k} p^k q^{n-k}$ $k=0, 1, \dots, n$		np	npq	$(pe^t + q)^n$

$P(X \geq 84) = \sum_{i=84}^{100} \binom{100}{i} p^i q^{100-i}$!!!
 $X \sim \text{Bino}(100, p)$

<u>Hypergeom</u> a of type 1 b of type 2 N of type 1	$P(X=k) = \frac{\binom{a}{k} \binom{b}{n-k}}{\binom{a+b}{n}}$	$n \left(\frac{a}{a+b} \right)$	$n \left(\frac{a}{a+b} \right) \cdot \left(1 - \frac{a}{a+b} \right)$ $\left(\frac{a+b-n}{a+b-1} \right)$
---	---	----------------------------------	---

NOTE: You have ∞ mixes of Events
 NOTE: Dists are related!

You can show this with mgf's!
 $Y = \sum X$
 $\left\{ \begin{array}{l} - \sum \text{Bern} = \text{Binomial} \\ - \sum \text{Geom} = \text{Neg Binomial} \\ - \text{Hypergeom} = \text{Binomial w/ smaller Var} \end{array} \right.$

NOTE: $\frac{a}{a+b} = p$
 in Binomial
 so $Y = \text{Bin}(n, p)$
 $\Rightarrow \text{Var} = n \left(\frac{a}{a+b} \right) \cdot \left(1 - \frac{a}{a+b} \right)$
 Same but larger Variance

<u>Geom</u> : # of trials "Until first success" *	$P(Z=k) = q^{k-1} p$ $P(Z > k) = P(\text{k Bernoulli failures in a row}) = q^k$ $P(Z \leq k) = 1 - (1-p)^k$	$1/p$	q/p^2	$M_Z(t) = \frac{pet}{1-qet}$ $\forall t < \ln(1/q)$
--	---	-------	---------	--

<u>NEG BIN</u> : the r th success in n Bern trials. $\sim \text{NegBin}(r, p)$ FFFSFS 6 trials until 2 nd success!	$P(W=k) = \binom{k-1}{r-1} p^r q^{k-r}$ $= \binom{k-1}{r-1} p^r q^{k-r}$ $\forall k = r, r+1, \dots$	$\frac{r}{p}$	$r \left(\frac{q}{p^2} \right)$	$\left(\frac{pet}{1-qet} \right)^r$
--	--	---------------	----------------------------------	--------------------------------------

Ex: W is # of trials until rth success.
 $W = r, r+1, \dots$

$$\int_0^1 \int_0^1 4xy \, dy \, dx$$

$$f_X(x) = \int_0^1 6xy^2 \, dy = 6x \int_0^1 y^2 \, dy = 6x \left[\frac{y^3}{3} \right]_0^1 = 2x$$

$$= 2x \quad 0 \leq x \leq 1$$

$$f(x) = 10x^2 = E[X^2] - 2$$

$$E[X] \int_0^1 x f(x) \, dx$$

$$f(x, y) = 10x^2 y \quad 0 \leq y \leq x \leq 1$$

$$E[C] = \int_{\mathbb{R}} C \cdot f(C) \, dC$$

$$E[X]$$

$$\text{Var}[X]$$

$$f_X(x) = \int_0^x f(x, y) \, dy = \int_0^x 10x^2 y \, dy = 5x^2 \left[\frac{y^2}{2} \right]_0^x = \frac{5}{2} x^3$$

$$f_X(x) = 5x^2 \cdot x = 5x^3 \quad 0 \leq x \leq 1$$

$$E[X] = \int_0^1 x \cdot 5x^3 \, dx$$

$$= 5 \left[\frac{x^5}{5} \right]_0^1 = 1$$

$$E[XY]$$

$$\int_0^1 \int_0^x xy f(x, y) \, dy \, dx$$

$$= \int_0^1 \int_0^x xy \cdot 10x^2 y \, dy \, dx = \int_0^1 10x^3 y^2 \, dy \, dx$$

Distrib

M4

M5

$$E[X^2] - 2\mu E[X] + \mu^2$$

$$\mu = E[X]$$

$$E[X^2] - 2\mu \cdot \mu + \mu^2$$

$$= E[X^2] - \mu^2$$

(N2)

$N(0,1)$ is kinda "new" distro.

pdf is $\phi(z) \equiv \frac{1}{\sqrt{2\pi}} e^{-z^2/2} \quad \forall z \in \mathbb{R}$

cdf is $\Phi(z) \equiv \int_{-\infty}^z \phi(t) dt \dots$ whatever...

usual gymnastics: $P(Z \leq a) = \Phi(a)$

$$P(Z \geq b) = 1 - P(\leq b) = 1 - \Phi(b)$$

$$P(b \leq Z \leq a) = \Phi(a) - \Phi(b)$$

Also, $\Phi(0) = 1/2$

$$\begin{aligned} \Phi(-b) &= P(Z \leq -b) \\ &= P(Z \geq b) \\ &= 1 - \Phi(b) \end{aligned}$$

$$P(-b \leq Z \leq b)$$

$$= \Phi(b) - \Phi(-b)$$

$$= \Phi(b) - [1 - \Phi(b)]$$

$$= 2\Phi(b) - 1 \quad !!$$

Awwwome! Now ask question about "within k standard devs of mean" ... :)

$$\begin{aligned} &P(\mu - k\sigma \leq X \leq \mu + k\sigma) \rightarrow \text{we } Z \equiv \frac{X - \mu}{\sigma} \\ &= P(-k \leq Z \leq k) = 2\Phi(k) - 1 \quad \Rightarrow X = Z\sigma + \mu \end{aligned}$$

ie, Prob. that any Norm RV is k std devs away from mean does not depend on mean or variance !!!

$$\begin{aligned} \text{Eg: } k=1 &\Rightarrow P = 0.6827 \\ k=2 &\Rightarrow P = 0.9545 \\ k=3 &\Rightarrow P = 0.9973 \end{aligned} \quad \left. \begin{array}{l} \text{TOYOTA'S } 6-\sigma \\ \Rightarrow 2\Phi(6) - 1 = 1.0000 \end{array} \right\}$$

let us define $\bar{X} \equiv \frac{1}{n} \sum_{i=1}^n X_i$ THIS IS the DEF of "SAMPLE MEAN"

then, $\bar{X} \sim N(\mu, \frac{\sigma^2}{n})$ (you can prove this...)

What's going on? Variance went down! and $E[\bar{X}]$ is constant!

This is Law of Large Numbers!

SIDE: $P(|\bar{X} - \mu| \leq 1) = P(-1 \leq \bar{X} - \mu \leq 1)$

so if $X \sim N(\mu, 16)$, then n has to be 62 ... for 95% chance ...
(can prove this...)

CLTeeee...! : If n is large, $\boxed{\bar{X} \approx N(\mu, \sigma^2/n)}$

$\Rightarrow$ The dist of X does not matter!!!

(usually, $n > 30$ works ... need fewer if symmetric dist...)

X is any RV. $Z_n \equiv \frac{\sum_{i=1}^n X_i - n\mu}{\sigma \sqrt{n}} = \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} = \frac{\bar{X} - E[X]}{\sqrt{\text{Var}[\bar{X}]}} \stackrel{d}{\sim} N(0, 1)$

means "the CDF of this"

$$P(A|B) = \frac{P(A \cap B)}{P(B)} \left\{ \text{def} = \right.$$

$$P(A|B) = P(A) \quad \{ \text{Independent} \}$$

$$M_x[t] = E[e^{tx}]$$

EXCEL or
CALC?

	mean	Var	PMF/PDF	CDF	MGF
DIST	$E[X]$	$\text{Var}[X]$			

Median (m): Solve $\sum 0.5 = F(m)$

$$\Rightarrow (x+y)^n = \sum_{i=0}^n \binom{n}{i} x^i y^{n-i} \quad \text{Binomial Theorem.}$$

$$2^n = (1+1)^n = \sum_{i=0}^n \binom{n}{i} (1)^i (1)^{n-i}$$

$$= \sum \binom{n}{i} \quad (?)$$

$$\binom{12}{1} = 12$$

$\Rightarrow$ Independent
and disjoint
not the same!

Hypergeom: WITHOUT Replac $\rightarrow \frac{\binom{a}{k} \binom{b}{n-k}}{\binom{a+b}{k}}$
Binomial: WITH Replac.

$$P(\text{ } \cancel{\text{ } } k \text{ } \text{ of } a\text{'s}) = \binom{n}{k} \left(\frac{a}{a+b}\right)^k \left(\frac{b}{a+b}\right)^{n-k}$$

If disjoint, you
know that if A happens
B cannot happen!

Assortments : $\binom{n_1 + n_2 + \dots + n_k}{n_1}$

A: sample space
|A|: size of sample space A

Arrangements : $\binom{n}{n_1, n_2, \dots, n_k} = \frac{n!}{n_1! n_2! \dots n_k!}$

Eg: MISSISSIPPI
(not words
to arrange)

$$= \frac{11!}{1! 1! 5! 1! 1!} = \frac{11!}{1! 2! 4! 2!} = 34,650$$

Cont. pdf Example

$$(0 < X < 1 \mid \frac{1}{2} < X < \frac{3}{2})$$

$$P(0 < X < 1 \text{ and } \frac{1}{2} < X < \frac{3}{2})$$

Intersection! Common area!

$$= P(\frac{1}{2} < X < 1)$$

$$= \int_{\frac{1}{2}}^1 f(x) dx$$

$$= \int_{\frac{1}{2}}^1 \frac{3}{2} f(x) dx$$

CDF properties

$$P(X \leq x) = F(x)$$



① Non-decreasing.
if $a < b$, $F(a) \leq F(b)$

note!

$$\textcircled{2} \lim_{x \rightarrow \infty} F(x) = 1$$

$$\lim_{x \rightarrow -\infty} F(x) = 0$$

Important!!

③ "Right continuity"

$$P(X > x) = 1 - F(x)$$

$$\textcircled{4} a < b \Rightarrow P(a < X \leq b) = F(b) - F(a)$$



$$\text{Var}(X) = E[X^2] - \mu^2$$

$$= E[X^2] - (E[X])^2$$

proof: $\text{Var}(X) = \text{"2nd Central moment" of } X$

$$\text{Var}(X) \equiv E[(X - \mu)^2]$$

$$= E[X^2 - 2\mu X + \mu^2]$$

$$= E[X^2] - 2\mu E[X] + \mu^2$$

$$\mu \equiv E[X] \therefore E[X^2] - 2\mu^2 + \mu^2$$

$$\boxed{\text{Var}(X) = E[X^2] - (E[X])^2}$$

$$E[aX + b] = aE[X] + b \rightarrow \text{"shift"}$$

$$\text{Var}(aX + b) = a^2 \text{Var}[X]$$

↓
does not "shift"

$$P(B) = P(A)P(B|A) + P(\bar{A})P(B|\bar{A})$$

CORRELATION!

$$\text{Corr}(X, Y) \equiv \frac{\text{Cov}(X, Y)}{\sqrt{\text{Cov}(X) \text{Cov}(Y)}} \equiv \frac{\sigma_{XY}}{\sigma_X \sigma_Y}$$

$\text{Corr}(X, Y) = \rho$ and you can show that $-1 \leq \rho \leq 1$

Formulas

whether or not independent,

$$V(X \pm Y) = \text{Var}(X) + \text{Var}(Y) \pm 2 \text{Cov}(X, Y)$$

if ind, $V(X+Y) = \text{Var}(X) + \text{Var}(Y)$

$$\rightarrow \text{Cov}(aX, bY+c) = ab \text{Cov}(X, Y)$$

MGFs again!

$$M_X(t) \equiv E[e^{tX}] \quad \text{use it!}$$

$$\text{and } E[X^k] = \left. \frac{d^k}{dt^k} M_X(t) \right|_{t=0}$$

$\rightarrow k^{\text{th}}$ derivative w.r.t t @ zero!

now! if X_i 's are independent (ONLY!) and $Y = \sum X_i$,

$$M_Y(t) = \prod_i M_{X_i}(t)$$

And! if X_i 's are indep and identically distributed,

$$M_Y(t) = [M_{X_i}(t)]^n \quad (\text{think about it!})$$

This is nice because now you can show $\sum \text{Bern}(x) = \text{Binom}(x)$ using MGF's! $\sum \text{Geom}(p) = \text{Neg Bin}(r, p)$

$$\text{Binom MGF} = (pe^t + q)^n$$

$$\text{Bern MGF} = pe^t + q$$

$$\text{Exp MGF} = \frac{\lambda}{\lambda - t}$$

$$\text{if } Y = aX + b, \quad \text{MGF}(Y) = e^{bt} M_X(at)$$

Side Uniform means

"Prob of picking any value in the range!"



Memoryless

let $Z \sim \text{Geom}(p)$
for +ve integers s, t ,

$$P(Z > s+t | Z > s) = P(Z > t) = q^t$$

This is unconditional!

"If something/event has not happ.
@ time s , prob. that it will happ.
@ time $(s+t)$, is same as prob. of
it happ. @ time t "

$$\text{Failure Rate func: } S(t) = \frac{f(t)}{P(X > t)}$$

continuous!

"Instantaneous
Rate of Death
after surviving after
time t "

$$S(t) = \frac{f(t)}{1 - F(t)}$$

HAND SIMULATION

Euler's Method

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

for small h ,

$$f'(x) \approx \frac{f(x+h) - f(x)}{h}$$

$$\Rightarrow f(x+h) \approx hf'(x) + f(x)$$

Eg: $f'(x) = (x+1)f(x)$, $f(0) = 1$
 $h = 0.01$, $x \in [0, 0.2]$

SIDE:

$$\begin{aligned} \text{MSE} &= \text{Var} + (\text{Bias})^2 \\ &= \text{Var}[X] + (E[X] - \mu)^2 \end{aligned}$$

→ We prefer low variance over bias

$$V(x) = E[X^2] - (E[X])^2$$

$$\text{Cov}(X, Y) = E[XY] - E[X]E[Y]$$

$$\text{Cov}(aX, bY) = ab \text{Cov}(X, Y)$$

$$\text{Cov}(X+a, Y+b) = \text{Cov}(X, Y)$$

$$\text{Cov}(aX+bY, cW+dV)$$

$$= \left. \begin{aligned} &ac \text{Cov}(X, W) + \\ &ad \text{Cov}(X, V) + \\ &bc \text{Cov}(Y, W) + \\ &bd \text{Cov}(Y, V) \end{aligned} \right\}$$

POISSON DIST

- $N(t)$ is "counting function" which counts # of things @ time
 - λ is avg. # of occurrences/unit time
 eg: $\frac{0.4}{in^3}$, $\frac{10}{min}$

Assume: ① Arrivals happen one/time then λ /time

② Arrival pattern is stationary

③ # of arrivals/time interval is constant independent



Eg: NOT POISSON: Customers at Restaurant

- ① Groups? GTFD!
- ② More at dinner time? GTFD!
- ③ Maybe not independent...

$U_1, U_2 \sim \text{Unif}(0,1)$ ♥
 $U_1 + U_2 \sim \text{Triangle}(0,1,2)$

$-\frac{1}{\lambda} \ln(u) \sim \text{Exp}(\lambda)$ *
 a) $-\frac{1}{\lambda} \ln(1-u) \sim \text{Exp}(\lambda)$

$N(\mu, \frac{\sigma^2}{n})$ CFA

$\bar{x} \sim N(\mu, \frac{\sigma^2}{n})$

goes down! when sample mean

0 if ind

$Cov(X,Y) = E[\cancel{E[X]E[Y]}]$
 $= E[(X - E[X])(Y - E[Y])]$

$Cov(X,X) = Var(X)$

$E[XY] - E[X]E[Y]$

$CORR(X,Y) = \frac{Cov(X,Y)}{\sqrt{V[X]V[Y]}}$

$-1 \leq \rho \leq 1$

$Cov(aX, bY) = ab Cov(X,Y)$

$P(A|B) = \frac{P(A \cap B)}{P(B)}$ $f(y|x) = \frac{f(x,y)}{f(x)}$

dep
 $= \frac{P(A)P(B)}{P(B)}$

$= P(A)$ indep

$E[X+Y] = E[X] + E[Y]$

$Var[X \pm Y] = Var[X] + Var[Y] \pm 2Cov(X,Y)$

$E[aX+b] = aE[X] + b$

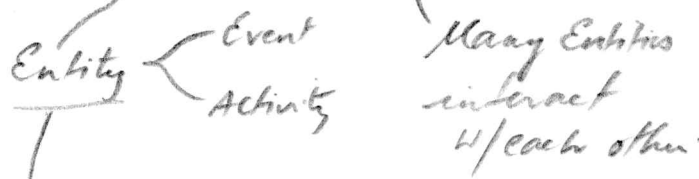
$V[aX+b] = a^2 V[X]$

$V[X] = E[X^2] - (E[X])^2$

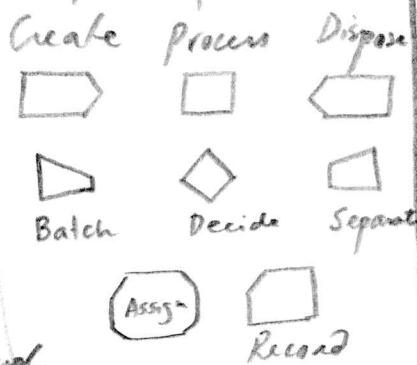
$E[E(Y|X)] = E[Y]$ CFA
 See Next
 Int'l
 test

① M5 ARENA!

⇒ The Process-Interaction Model



These flow through modules or block



You get Templates

The most used one is the Basic Process Template

Modules
Spreadsheets (for entities, their attributes, etc)

The **PROCESS** module

- Grab servers, use, let them go

* - sets up a queue automatically!

Types → Seize : self-service

→ S/Delay/Release :

→ S/D : Eg. Hospital beds.
You do a "release" after you clean up!

* Will deadlock if no RELEASE later!!

→ D/R : Release prev. seized server!

You get an "auto-queue" called

{ process name }. Queue

Process = Resource + Queue

SPREADSHEETS

Resource Spreadsheet

Capacity in the # of Resources available! not the # the cust. requests!

Resources automatically set up by PROCESS module!

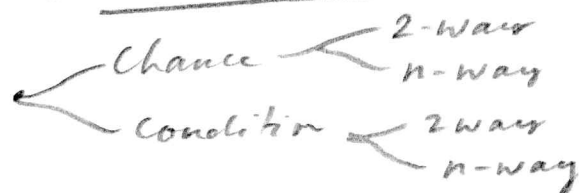
Schedule Spreadsheet : Change Capacity

of Resources (like if you want servers to take smoke breaks)
Change Arrivals of customers!



Decide Module

Simple!



Assign Module

height weight

Allows you to assign Attributes (so you can filter and decide where to route them later!)

Have to be numerical!

Can use pictures! ♡

Attributes are specific to entity!

Variables are global!!

Things go to Spreadsheets



Attr



Var



Entity

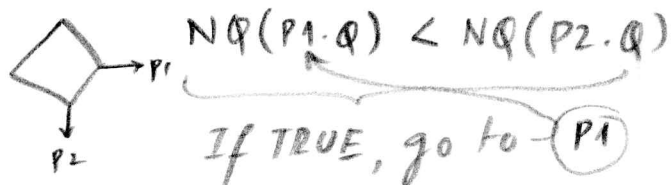
Create generic things/entities

- ↳ Assign properties/attrs to them
- ↳ Process based on these props

Internal Variables

These are global! Affect the Entire Simulation!

- $TNOW$: Current Sim. time
- $NR(\text{Resource})$: # of res. servers (not total!) currently working
- $NQ(\text{Proc. Queue})$: # of cust. in queue
- $\{\text{Module}\}.\text{NumberOut}$: # of cust. who have exited module.



BATCH/SEP/RECORD Modules
(last 2 basic proc. templ)

BATCH

Eg. for non-homo poisson.

Group customers to "super custom"

SEP

Split batch

Duplicate customers!

REC

just record stats... Simple

B

S

→ "Permanent" ⇒ the 'SuperCustomer' loses all individual attributes

RUN SETUP

usual, commonsense stuff

- ↳ Replication?
- ↳ "Warmup time"
- ↳ Accumulate stats? (Initialize)
- ↳ Speed
- ↳ Terminating condition?
- ↳ "Category Overview" gives summary
- ↳ Turn off graphics? Batch Mode
FAST!

ADVANCED ARENA

- Attributes are "memorized" by incoming customer. Use ASSIGN!
- Logical Exps are in (-----)
 - { If TRUE → yield 1 } actual
 - { Else → yield 0 } IN+ 's!
- Use "Fake Customers"
 - ↳ Don't care about their serv. times
 - ↳ Use those to "stress test" or generate breakdowns
- There's ADV. PROC. TEMPL
 - ↳ Ships with its own separate SEIZE DELAY RELEASE module
 - ↳ Now can do S/ASSIGN/D/R
 - ↳ Asymmetric SEIZE/REL.
 - ↳ Complex rules (of course) for S/R

② M5

Can use "Fake Customer" to

Cause fail. (OR) Use Resource Spdsht (Basic Templ)

* Use Failure Spdsht (Adv Templ)

How to fail

"after x people
come in, fail with
downtime"

Count (# of)

Time (duration of)

Downtime (Distribn)

Fail RULES

Serv time 10 min
Repair time 1 hr

IGNORE - Serve cust. then
reduce repair time

Eg finish @ 60 min
then finish repair = 50 min

WAIT - Serve Cust. then
delay repair time

Eg finish @ 70 min
repair (reptime + cust time)

PREEMPT - Stop Everything,
repair, then serve cust.

Eg: Cust gets served @ 70 mins.

⇒ You can SEIZE with all
kinds of rules.

{ Cyclic
Random
Most Resources in set
.... many more.

One or Two lines?



(a)



(b)

(a) is almost
always better!

SETS: Common-sense with Resources.

Make a set of Resources. Each
element in the set can have
different abilities → Resource Set!

⇒ You have to release the same sem!

You do this by remembering
the index of the agent.

- SMARTS files are tutorials
by Rockwell (not demos!)

- Re-entrant Queues in
Tutorials onillate (and
get larger and larger).

⇒ "Adv. Manufac. Cell Demo"
{ Need the "Adv. Transfer Template"
Makes 3 kinds of part.

Each one has a different path,
thru the system → A Sequence

Each can visit diff. service
stations in any order, any # of
times ($S1 \rightarrow S2 \rightarrow S1 \rightarrow S3 \rightarrow S4 \rightarrow S2$)

→ Need a sequences spreadsheet

→ Need ROUTE STATION ENTER LEAVE
modules (from Adv. Xfer template)

→ Need Adv. sets (of sequences)

(Note: You can make sets of
anything! Anything at all!!)

→ Use the ROUTE block to define
sequences (duh!) Start
Go to ~~Route~~
Go to ---
in a spreadsheet

↓
"Go from X to Y" (that's it)
in Z time

↓
Defined in Seq. Spreadsheet!

→ Use STATION to define destinations.

→ Transporter takes cust. between
stations. Resource must be movable!
Forklift, Cab, etc.

→ Conveyer. Belts, elevators, etc.
(Common sense plz)

①

MODULE 6

Goal: Unif (input) → Exp RV → Norm RV (output)

Unif RV's are the key to RV generation - Derivables

- Output appears to be iid
- Fast
- Reproduce sequence (loop thru, deterministic)

Shitty Generators

① Toss a coin. Die. Ok but storage is a problem. We are thinking about billions of things!

② Book of Random Digits: In book, not random anymore! Very small.

③ Mid-Square Method: Newman!

$$X_0 = 6632 \rightarrow \text{Square} \rightarrow 43983424$$

$$X_1 = 9834 \rightarrow \text{Square} \rightarrow 96707556$$

$$X_2 = 7075 \rightarrow \dots$$

$$R_i = X_i / n \quad \text{eg: } X_2 = \frac{7075}{10,000} = 0.7075$$

Problems? ① +ve serial correlation
② you can get 0003!
"degenerate"!

④ Fibonacci and "Additive Congruential Generators"

$$X_i = (X_{i-1} + X_{i-2}) \bmod m$$

$$a \equiv b \bmod m \rightarrow a \text{ is 'congruent' to } b$$

Problems? ① Small #'s follow small #'s
② Cannot get stuff in between!
eg You can't get $X_2 < X_3 < X_1$, should be able to do this w. prob $\frac{1}{3}$

Linear Congruential Generators LCG

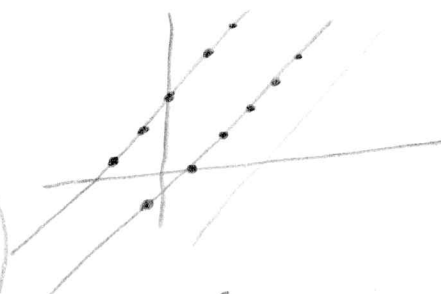
$$\text{General Form: } X_i = (aX_{i-1} + c) \bmod m$$

$$\text{and } R_i = X_i / m$$

You have to pick ① ③ and ④ for
→ Good "statistical property"
→ Good cycle-length.

→ if the cycle-length == m , we have a "full-period generator".

PLANES! LCG's generated #'s fall in lines/planes: Plot (X_{i-1}, X_i)



also, $X_i = (5X_{i-1} + 2) \bmod 8$
if this starts cycling before
so you have to pick a, c, m properly!

→ Desert Island Generator

$$X_i = 16807 X_{i-1} \bmod (2^{31} - 1)$$

(multiplicative $\because$ no $c : c=0$)

→ Full-cycle if you don't start at zero!

Algorithm: There are magic #'s involved!

$$① X_i = 16807 \left[X_{i-1} - 127,773 \times \text{Math.floor} \left(\frac{X_{i-1}}{127,773} \right) \right] - 2836 \left[\text{Math.floor} \left(\frac{X_{i-1}}{127,773} \right) \right]$$

if $X_i < 0$, $X_i \leftarrow X_i + (2^{31}-1)$

and finally, $R_i = \frac{X_i}{2^{31}-1}$

WHERE THINGS CAN GO TO SHIT

- Not full-period: $(4X_{i-1} + 2) \bmod 8$
"ONLY EVENS!!"
- Full-period but not random

Eg: $X_i = (X_{i-1} + 1) \bmod 8$

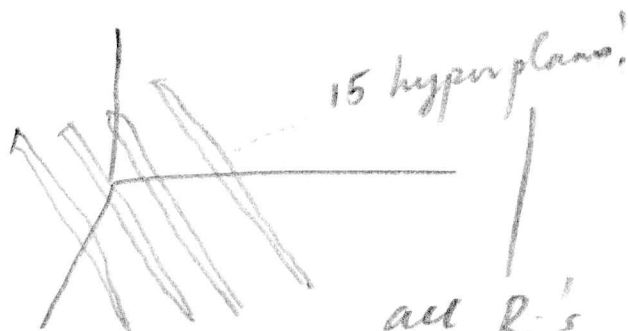
W/ $X_0 = 0$, $X_1 = 1$ $X_2 = 2$ $X_3 = 3 \dots$ LOL

- m is small $\Rightarrow$ quick cycling (< 2 billion)

- m is big? Problems like 3D hyperplanes in RANDU

~~$X_i = 65539 X_{i-1} \bmod 2^{31}$~~

if you plot (R_{i-2}, R_{i-1}, R_i)



15 hyperplanes

all R_i 's fall on these!

② If $X < 0$, $X_i \leftarrow X_i + 2147483647$

③ $R_i = X_i \times (4.656612875) \times 10^{-10}$

TAUSWORTHE Generator

Provide a seq of Binary Digits B_n

$$B_i = \left(\sum_{j=1}^q C_j B_{i-j} \right) \bmod 2$$

where $C_j = 0$ or 1

This is nice for computers...

$$B_i = (B_{i-r} + B_{i-q}) \bmod 2$$

$$B_{i+r} = B_{i-r} \text{ XOR } B_{i-q} \quad (0 \leq r \leq q)$$

EXAMPLE $r=3$, $q=5$. Initialize $B_1, \dots, B_5 = 1$

$$\Rightarrow B_i = B_{i-3} \text{ XOR } B_{i-5} \quad \forall i \geq 5$$

Eg: $B_6 = B_{6-3} \text{ XOR } B_{6-5}$
 $= B_3 \text{ XOR } B_1 = 0$

$B_7 = B_4 \text{ XOR } B_2 = 0 \dots$

XOR		
	0	1
0	0	1
1	1	0

"Exclusive" part of XOR

Notes: $B_i = 0$ if $B_{i-r} = B_{i-q}$

$B_i = 1$ if $B_{i-r} \neq B_{i-q}$

and you have to initialize q B 's

Get Unifs by $\frac{\text{kbits in bank}}{2^k}$

Eg: $1111_2 = \frac{1111_2}{2^4}$

② M6 Generalization of LCG's

Simple one: $X_i = \left(\sum_{j=1}^q a_j X_{i-j} \right) \bmod m$

Can (not guaranteed!) have very large periods: $(m^q - 1)$

Fibonacci is a special case tho

Combine: $X_i, \dots, Y_i$ make

$$Z_i = (X_i + Y_i) \bmod m$$

then shuffle: Set $Z_i = X_i$ or Y_i

$\Rightarrow$ Properties are very hard to prove!

L'Ecuyer (1999)

Nasty but gives cycle-length of 2^{191} (approx)

Mersenne Twister:

Cycle length of $(2^{19,937} - 1)$

W.T.F!!!

$\Rightarrow$ Typically don't need PRNG's larger than 2^{100}

$\Rightarrow$ "k-tuple" means k # of sequential or consecutive random #s R_i
($R_i, R_{i+1}, R_{i+2}, \dots$)

So now what to do when you have hyperplanes?

You want to fill a space w/ as many hyperplanes as possible to make it look like "real" noise.

$\uparrow$ # of planes $\downarrow$ dist bet adj k -tuples

$\downarrow$ dist. between planes

so, "Maximize the minimum #"

"Min the max distance"

"Max the min dist bet adj k -tuples"

Serial correlation of PRNG's

$$\text{Corr}(R_1, R_2) \leq \frac{1}{a} \left[1 - \frac{6c}{m} + 6 \left(\frac{c}{m} \right)^2 \right] + \left(\frac{a+b}{m} \right)$$

a is something m 's upper-bound has to be very small (~ 2 Billion)

*Greenberg, 1961

STATISTICAL TESTS

Two kinds:

Goodness of fit
"are they $\sim U(0,1)$?"

Independence tests
"are they $\sim$ Independent?"

H_0 : They (PRN's) are iid
(null) (innocent until guilty!)

① (more important!)
P(reject H_0 | H_0 is true)
0.05, 0.01, etc. Type I error

② P(accept H_0 | H_0 is false)
Type II

χ^2 gof test! (There are many more)

Test $H_0: R_1, R_2, \dots \sim U(0,1)$

"Slot" R_n 's into k cells

→ We want to see if the Exp. Val. of R_n 's in each cell matches the Actual Value of R_n 's dist across k cells.

$O_i \sim \text{Bin}(n, \frac{1}{k})$

$E_i = \frac{n}{k} \because \text{Binom}$
 $= E[O_i]$

prob. that an R_n will land in cell k!

→ Now use χ^2 . Test statistic is

$$\chi^2_0 = \sum \frac{(O_i - E_i)^2}{E_i}$$

Big Value $\Rightarrow$ bad fit "

Now, Reject H_0 iff $\chi^2_0 > \chi^2_{\alpha, k-1}$

Fail to Reject if

$$\chi^2_0 \leq \chi^2_{\alpha, k-1}$$

→ Because of mild correlation. If R_n is in one cell, it cannot be in another! $\therefore k-1$ dof

PROBLEM: n and k

can be HUGE.

to approx. $\chi^2_{\alpha, k-1} \approx (k-1) \left[1 - \frac{2}{9(k-1)} + Z_{\alpha} \sqrt{\frac{2}{9(k-1)}} \right]^3$

→ There are other gof's < Kolmo - Smirnov, Anderson - Darling

EG: $n=1000$, $k=5$ cells/intervals.

Interval	[0, 0.2]	(0.2, 0.4]	(0.4, 0.6]	(0.6, 0.8]	(0.8, 1.0]
E_i	1000/5	200	200	200	200
O_i	178	292	201	172	162

Compute χ^2_0 and $\chi^2_{0.05, 5-1}$ and fail to reject (or not)

Independence Tests

HT HT HT HT ... (-ve correlation)
HH HH TTTT ... (+ve correlation)
HTHTTHTH ... (okay??)

→ There are "runs": series of III^y observations
There are many run tests!

① Up and Down (FUN!) $n=6$

0.41 0.68 0.89 0.84 0.74 0.91
+ + + - - +

Let A be a RV $\rightarrow$ # of runs in n samples.

A = 3

CRAZY! If n is large (≥ 20) R_n 's are ind,

$$A \approx N\left(\frac{2n-1}{3}, \frac{16n-29}{90}\right)$$

$E[A] \quad \text{Var}[A]$

③ M6 Now the test-stat!

$$Z_0 = \frac{A - E[A]}{\sqrt{\text{Var}(A)}}$$

Reject if $|Z_0| > Z_{\alpha/2}$
 $\rightarrow$ modulus!!
 $\rightarrow$ reverse lookup!

② Above and Below Mean (which is 0.5)

if R_i is ≥ 0.5 , add +
 if R_i is < 0.5 , add -

$$B \approx N\left(\frac{2n_1 n_2}{n} + \frac{1}{2}, \frac{2n_1 n_2 (2n_1 n_2 - n)}{n^2 (n-1)}\right)$$

$n = n_1 + n_2$
 # more than 0.5
 # less than 0.5

Same test stat! $Z_0 = \frac{B - E[B]}{\sqrt{\text{Var}(B)}}$

Reject if $|Z_0| > Z_{\alpha/2}$

M7 RANDOM VAR GEN

Use the $\text{Unif}(0,1)$ dist to generate "realizations" from all other dist.

The key thing there is the Inv. Trans. Thm

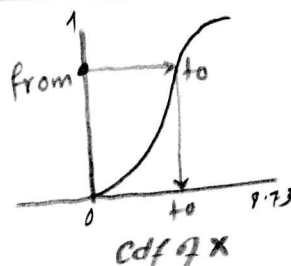
$\rightarrow$ if $F(x)$ is CDF of X , $F(X) \sim U(0,1)$

So you basically solve for

$$F(x) = u$$

eg: $F(x) = 1 - e^{-\lambda x}$ Exp. dist.

$$F(x) = u \Rightarrow x = -\frac{1}{\lambda} \ln(u) \text{ or } -\frac{1}{\lambda} \ln(1-u)$$



Let's try Weibull (cont.)

$$F(x) = 1 - e^{-(\lambda x)^\beta}$$

Set $F(x) = u \Rightarrow x = \frac{1}{\lambda} [-\ln(1-u)]^{1/\beta}$
 or $\frac{1}{\lambda} [-\ln(u)]^{1/\beta}$

(generalizes Exponential)

$\rightarrow$ What about separate cars?

Eg: $\text{Tri}(0,1,2)$

$$f(x) = \begin{cases} x & 0 \leq x \leq 1 \\ 2-x & 1 \leq x \leq 2 \end{cases}$$

$$F(x) = \begin{cases} x^2/2 & 0 \leq x \leq 1 \\ 1 - \frac{(x-2)^2}{2} & 1 \leq x \leq 2 \end{cases}$$

to make separate U 's!

$$\cancel{U = \sqrt{2x}} \quad X = \sqrt{2U} \quad U < \frac{1}{2}$$

$$\text{or } X = 2 - \sqrt{2(1-U)}$$

corresponds to $0 \leq x \leq 1$

$\Rightarrow$ Take care of $\pm$!

x has to be between 1, 2 $U \geq \frac{1}{2}$

$\Rightarrow$ You cannot replace U with $(1-U)$

$\rightarrow$ What about Normal Dist?

$\Phi^{-1}(\cdot)$ has no closed form!!

STANDOR

$\rightarrow$ Do a table lookup.

if $U = 0.975$, $Z = \Phi^{-1}(U) = 1.96$

Can use the same for
Normal (not Std Normal)

Eg: Generate $N(3, 16)$ from U

$$X = \mu + \sigma Z = 3 + 4Z$$

$$Z = \Phi^{-1}(U)$$

$$\Rightarrow X = 3 + 4\Phi^{-1}(U)$$

so if $U = 0.59$,

$$X = 3 + 4\Phi^{-1}(0.59) = 3.91$$

lookup!
in reverse!

DISCRETE DISTS

• Bernoulli

x	$P(X=x)$	F(x) $V(0,1)$
1	p	$[0, p]$
0	$1-p$	$(p, 1]$

so if $p = 0.75$. Let the
Unif fall between 0 and p ,
then $X=1$ else 0
ie, if $U_i \leq p$, $X=1$

• other

x	$P(X=x)$	$F(x)$	$V(0,1)$
-1	0.6	0.6	$[0, 0.6]$
2.5	0.3	0.9	$(0.6, 0.9]$
4	0.1	1.0	$(0.9, 1.0]$

if $U = 0.63$, $X = 2.5$

• Geometric $f(x) = q^{1-k} p$ | $F(x) = 1 - q^k$
of trials until
 k^{th} success

$$\text{solving, } X = \left\lceil \frac{\ln(U)}{\ln(1-p)} \right\rceil \sim \left\lceil \frac{\ln(1-U)}{\ln(1-p)} \right\rceil$$

$\Rightarrow$ also count Bern(p) until successful

$$U = 0.73 \quad 0.81 \quad 0.12$$

and $p = 0.3$, then count trials until

$$U_i \leq p$$

so here, $X=3$

• Poisson: kinda the same idea
as the "other" one.

$$\text{EG: } X \sim \text{Pois}(2) \quad f(x) = \frac{e^{-2} 2^x}{x!}$$

x	$f(x)$	$F(x)$	$V(0,1)$
0	a	a	$[0, a]$
1	b	$a+b$	$[a, a+b]$
2	c	$a+b+c$	$[a+b, a+b+c]$

so if you spin the wheel and see
 U between a and $a+b$, it means
that $X=1$. Congrats!

EMPIRICAL DISTS

You don't know which dist the
RV's came from. Say they are
continuous.

Use an Empirical CDF

$$\hat{F}_n(x) = \frac{\# \text{ of } X_i \text{'s} \leq x}{n}$$

step function.

4) M6-M7

Glinenko-Cantelli Lemma

says $\hat{F}_n(x) \rightarrow F(x)$ as $n \rightarrow \infty$

so you can use it in a cont. context even tho it is a step function.

ORDERED POINTS.

if $X_1 = 2$ $X_2 = 14$ $X_3 = 1$.

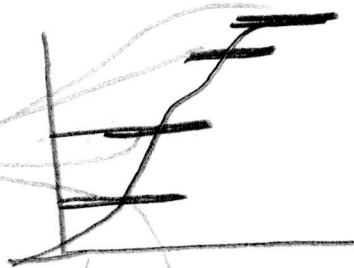
then $X_{(1)} = 1$ $X_{(2)} = 2$ $X_{(3)} = 14$

put subscripts in parens

ARENA: Can use DISC and CONT to generate RV's for Empirical

OK SO,

This is $\hat{F}_n(x)$ for ordered points!



True but unknown CDF!

Step function which we can make continuous. With a shit Equation that does Linear Interpolation of $F(x)$

Interpolated,

$$F(x) = \begin{cases} 0 & \text{if } x < X_{(1)} \\ \frac{i-1}{n-1} + \frac{x - X_{(i)}}{(n-1)[X_{(i+1)} - X_{(i)}]} & \text{if } X_{(i)} < x < X_{(i+1)} \\ 1 & \text{if } x \geq X_{(n)} \end{cases}$$

CONVOLUTION: (ude Desert Island) (doubt)

if $Y = \sum U_i$, $Y \approx N(\frac{n}{2}, \frac{n}{12})$ ($\because$ CLT) NOW MAKE STD NORM!! $Z = \frac{Y - n/2}{\sqrt{n/12}}$

That shit. set $F(x) = U$

set $P = (n-1)U$

set $I = \lceil P \rceil$

now generate $X = X_{(I)} + (P - I + 1) \frac{X_{(I+1)} - X_{(I)}}{P - I + 1}$

Random Starting point

Eg: $X_{(1)} = 1$ $X_{(2)} = 4$ $X_{(3)} = 6$

if $U = 0.73$, find X .

$$X = X_{(I)} + (P - I + 1) \frac{X_{(I+1)} - X_{(I)}}{P - I + 1}$$

$$P = (3-1)(0.73) = 1.46$$

$$I = \lceil 1.46 \rceil = 2$$

$$\Rightarrow X = X_{(2)} + (1.46 - 2 + 1) \frac{X_{(3)} - X_{(2)}}{1.46 - 2 + 1}$$

$$\Rightarrow X = 4.92$$

SET $F(x) = U$ and SOLVE

Convolution

About sums, really.

$$\sum_{i=1}^n \text{Bern}(p) = \text{Binom}(n, p)$$

Same way, Eg. if $Y \sim \text{Bin}(3, 0.4)$

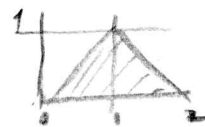
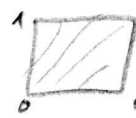
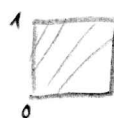
and $U_1 = 0.63$, $U_2 = 0.17$, $U_3 = 0.81$

$$\therefore Y = 0 + 1 + 0 \quad \left[\begin{array}{l} \text{if } \leq 0.4, \\ x = 1 \end{array} \right]$$

$$\Rightarrow Y = 1$$

You already know this:

$$U(0,1) + U(0,1) = \text{Tri}(0,1,2)$$



→ You already know this.

$$\sum_{n=1}^{\infty} \text{Exp} = \text{Erlang}_n$$

So yeah...

$$Y = \sum_{i=1}^n X = \sum_{i=1}^n \left[-\frac{1}{\lambda} \ln(U) \right] = -\frac{1}{\lambda} \ln \left[\prod_{i=1}^n U \right]$$

$$\rightarrow \sum_{n=1}^{\infty} \text{Geom} = \text{Neg Bin}$$

$$\rightarrow \sum_{n=1}^{\infty} Z = \chi^2$$

→ if $X \sim \text{Cauchy}$, $\bar{X} \sim \text{Cauchy}$

This is $\because$ Cauchy violates CLT

so Adding Dists up is called
CONVOLUTION ☺

ACCEPTANCE REJECTION Method

① Inv transform

② Convolution

③ Acceptance-Rejection.

Can use for stochastic as well

→ Works. Always. Somehow.

→ So it's a bit difficult.

→ Von Fucking Neumann (1951)

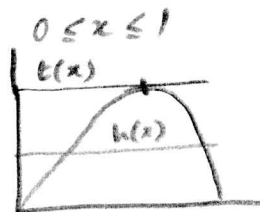
TODO: Write it down...

→ Mean # of trials is c

→ # of trials until success is $\text{Geom}(\frac{1}{c})$

Eg: pdf of RV in

$$f(x) = 60x^3(1-x)^2$$



This cannot be inverted analytically,

$$F^{-1}(U) = X \text{ — not possible.}$$

The players: $f(x)$ $h(x)$ $t(x)$ $g(x)$

$f(x)$: The pdf.

$t(x)$: "Majorizer": job is to be always larger than $f(x)$

$$h(x) = \frac{t(x)}{c} \quad c = \int_{-\infty}^{\infty} t(x)$$

$$g(x) = \frac{f(x)}{t(x)} \quad \text{if } U \leq g(x)$$

so for a given U , LOOP until

do calc. max @ 2.076 "Inefficient"

so make a dumb maximizer

$$f(x) = 2.076! \text{ A line!}$$



$$\Rightarrow c = \int_0^1 t(x) = 2.076 \quad \Rightarrow g(x) = \frac{60x^3(1-x)^2}{2.076}$$

$$\Rightarrow h(x) = t(x)/c = 1 \quad \text{how if } x = 0.25 \text{ and } U = 0.13, U \leq g(x) \text{ ACCEPT}$$

⇒ Use this to Generate POISSON?

Did this before... but use AR

REVIEW

$X = n$ if we see exactly n arrivals in a time unit

A_i is the inter-arrival time in a $\text{Pois}(\lambda)$ process

It is exponential!!

So, if $X = n$ (5) MC-7

$$\Rightarrow \sum_{i=1}^n A_i \leq 1 < \sum_{i=1}^{n+1} A_i$$

" n^{th} arrival happens by time 1"

" $(n+1)^{\text{th}}$ arrival happens after time 1"

and A_i is $\text{Exp}(\lambda)$

$$\Rightarrow \sum_{i=1}^n \left[-\frac{1}{\lambda} \ln U_i \right] \leq 1 < \sum_{i=1}^{n+1} \left[-\frac{1}{\lambda} \ln U_i \right]$$

$$\Rightarrow \left[\prod_{i=1}^n U_i \geq e^{-\lambda} > \prod_{i=1}^{n+1} U_i \right]$$

TLDR: multiply U_i until it is smaller than $e^{-\lambda}$

Eg: $e^{-\lambda} = 0.1349$

n	U_{n+1}	$\prod U_i$	Stop?
0	0.3911	0.3911	NOPE
1	0.94		NOPE
2			NOPE
3		0.1304	YEP

Expected # of U_i needed?

$$E[X+1] = \lambda + 1$$

➔ If $\lambda \geq 20$, $\frac{X - \lambda}{\sqrt{\lambda}} \approx N(0, 1)$

How to use:

$$X = \max(0, \lfloor \lambda + \sqrt{\lambda} Z + 0.5 \rfloor)$$

make this up. Remember: Norm has no closed form for CDF !! have to do table lookup.

⇒ Use to Generate Unif(a, b)?

Eg: Generate $\text{Unif}(\frac{2}{3}, 1)$

ROLL! Let $U_1 = 0.16$

NOPE! ROLL until $U_i \geq \frac{2}{3}$

Composition Method

Useful for "mixes" of RV's which are not the same dist...

$$F(x) = \sum_{j=1}^{\infty} F_j(x) \cdot p_j \quad \left| \sum_{j=1}^{\infty} p_j = 1 \right.$$

"With p_j prob., pick $F_j(x)$ "
it's that easy!

That Pesky Normal Dist !!

But! Use Box-Muller!

Take U_1 and U_2 to make Z_1, Z_2

$$Z_1 = \sqrt{-2 \ln(U_1)} \cos(2\pi U_2)$$

$$Z_2 = \sqrt{-2 \ln(U_1)} \sin(2\pi U_2)$$

⇒ Must be in Radians!!!

and $Z_1, Z_2 \sim N(0, 1)$!!

⇒ Can use this to generate distribution trivia.

POLAR Method as an alternative to Box-Muller.

take $U_1, U_2 \sim U(0,1)$

let $V_i = 2U_i - 1$ (center it)

let $W = V_1^2 + V_2^2$

Loop: If $W > 1$, regenerate U_1, U_2 ^{reject}

else, $Y = \sqrt{\frac{-2\ln(W)}{W}}$

and $Z_i = V_i Y$ ☺

The usual, $G(Y) = V$
 $\Rightarrow Y = G^{-1}(V)$

$$\text{and } Y = F^{-1}\left[1 - (1-U)^{1/n}\right]$$

(Eg) $X_1, \dots, X_n \sim \text{Exp}(\lambda)$

get the min using one U

$$G(y) = 1 - [1 - (1 - e^{-\lambda y})]^n$$
$$= 1 - e^{-\lambda y n}$$

this is cdf of $\text{Exp}(n\lambda)$!

So INVTTRANS and $Y = -\frac{1}{n\lambda} \ln(U)$

Order Statistics (Random stuff)

lets say you want

$$Y = \min\{X_1, \dots, X_n\}$$

(any dist w cdf $F(x)$
has cdf $G(y)$)

Can you use just ONE U ?

$$\text{So, } G(y) = P(Y \leq y) = 1 - P(Y > y)$$

$$\Rightarrow G(y) = 1 - P(\min(X_i) > y)$$

if the min of X is $> y$, then all are $> y$ (duh)

$$\Rightarrow G(y) = 1 - [P(X_i > y)]^n$$

$$\Rightarrow G(y) = 1 - [1 - F(x)]^n$$

$$\rightarrow \sum_{i=1}^n Z_i^2 = \chi^2(n)$$

$$\rightarrow \text{if } Z \sim N(0,1) \quad Y \sim \chi^2(n)$$

$$\text{then } \frac{Z}{\sqrt{Y/n}} \sim t(n) \quad | \quad t(1) \sim \text{Cauchy}$$

$$\rightarrow X \sim \chi^2(n) \quad Y \sim \chi^2(m)$$

$$\text{then } \frac{(X/n)}{(Y/m)} \sim F(n, m)$$

$$\rightarrow \frac{Z_2}{Z_1} \sim \frac{N(0,1)}{N(0,2)} \sim \text{Cauchy} \sim t(1)$$

$$\rightarrow Z_1^2 + Z_2^2 \sim \chi^2(1) + \chi^2(1) \sim \chi^2(2)$$
$$\sim \text{Exp}(\lambda/2)$$

⑥ M6-7

Markov Chain

→ You have a state-transition matrix
→ So you go from state → state
based on this matrix.

This is easy to
simulate w/ U's

	Yesterday rain	Today rain	Sun
rain		0.7	0.3
Sun		0.4	0.6

EG start!

Suppose it rains on Monday,
make rest of week.

	$P(R X_{i-1})$	U_i	$U_i < P_{ij}$	rain/Sun
M	Today			
T	0.7	0.62	YES	R Y
W	0.7	0.03	Y	R Y
Th	0.7	0.77	N	S Y
F	0.4	0.91	N	S

That's really it. Each row
is today based on Yesterday

Poisson Process

① When you want to simulate
∞ arrivals at rate λ

$$T_0 = 0, \quad T_i \leftarrow T_{i-1} + A_i$$

$$T_i \leftarrow T_{i-1} - \frac{1}{\lambda} \ln(U)$$

② When you want a
guaranteed # of arrivals (n)
@ rate λ between $[a, b]$

This is weird but there is a
nice theorem: ① Take n $Unif(a, b)$
② Put them in order
③ Done!

So, generate $U_1, U_2, U_3, \dots, U_n$
Sort $U_{(1)}, U_{(2)}, U_{(3)}, \dots, U_{(n)}$

adjust $T_i = a + (b-a)U_i$



Non Homo. Poisson

Arrival
rate is λ
not constant

→ Instead of λ , you get a function
that describes how λ changes.

Eg: $\lambda(t) = t^2$. Find the Prob. of
Exactly 4 arrivals bet. ~~times~~
times 1 and 2.

Generally, $N(s+t) - N(s) \sim \text{Pois}\left(\int_s^{s+t} \lambda(t) dt\right)$

so $N(2) - N(1) \sim \text{Pois}\left(\int_1^2 t^2 dt\right) \sim \text{Pois}(7/3)$

so $P(X=4) = \frac{e^{-\lambda} (\lambda)^n}{n!} = \frac{e^{-7/3} (7/3)^4}{4!}$

OK So now how do you
generate non-homog. Poisson
arrivals?

OK so that's fine Math-wise
need computers. And if you
have wonky $\lambda(t)$.

So use Thinning Algorithm

Accept
reject
method

→ First, get the max arrival rate λ^*
using calculus or something

→ You will need two Unifs

$U \sim U(0,1)$: Use this to generate
"potential" inter arrival times
using $(-\frac{1}{\lambda} \ln(U))$

$V \sim U(0,1)$. Use this to accept
or reject the arrival time
(NOT INTER! SUM THINGS!)
if $V \leq \frac{\lambda(t)}{\lambda^*}$, accept

SUM of all inter arrivals upto that
point! See Examples.

CHEATSHEET! EXAMPLE!!

BROWNIAN MOTION

"Prob the most imp. Stochastic
process out there"

Brown
+ Einstein
+ Wiener

→ Actually used in
Queueing Theory, Finance, etc

① Start at $W(0) = 0$

② $W(t) \sim N(0, t)$

③ $\{W(t), t \geq 0\}$ has "stationary"
and "independent"
increments.

Stationary : $W(t+h) - W(t)$
depends only on h

Independent : $a < b < c < d$
 $W(d) - W(c)$ indep. of
 $W(b) - W(a)$

"Generalization of a Random Walk

How to get it? Take $X_n \sim N(0,1)$

$$\left\{ \frac{1}{\sqrt{n}} \sum_{i=1}^{\lfloor nt \rfloor} X_i \xrightarrow{d} W(t) \quad n \rightarrow \infty \right.$$

means "dist converges to"

So if you set $t=1$, → CLT

$$W(1) \sim N(0,1)$$

This is Lévy's CLT

(7) M6-7. So the idea is to generate some noise in a system $\rightarrow W(t)$

- It is successive, $W(0) \rightarrow W(1) \rightarrow W(2) \rightarrow W(3)$

$\rightarrow$ So how to generate?

$$\underbrace{W\left(\frac{i}{n}\right)}_{\text{new}} = \underbrace{W\left(\frac{i-1}{n}\right)}_{\text{previous}} + \frac{Y_i}{\sqrt{n}}$$

ok. What is Y_i ? $\begin{cases} \pm 1 \text{ w prob } \frac{1}{2} \\ \text{or} \\ N(0,1) \end{cases}$

n needs to be large, ok?

Props: ① Too squiggly/sharp
NO DERIVATIVES! WTF!

② $\text{Cov}(W(s), W(t)) = \min(s, t)$

③ $\int_0^t W(t) dt \sim N(0, \frac{1}{3})$

Bridges

④ $B(t)$ means $\underbrace{W(0)=W(1)=1}_{\text{"Conditioning"}}$

⑤ $\text{Cov}(B(s), B(t)) = \min(s, t) - st$

⑥ $\int_0^t B(t) dt \sim N(0, \frac{1}{12})$

GEOM BROWNIAN

$$S(t) = S(0) e^{(\mu + \frac{\sigma^2}{2})t + \sigma W(t)}$$

$t \geq 0$

again,

$$S(t) = S(0) \exp\left[(\underbrace{\mu}_{\text{Initial price!}} - \underbrace{\frac{\sigma^2}{2}}_{\text{"Drift"}})t + \underbrace{\sigma W(t)}_{\text{"Volatility"}}$$

$\frac{\sigma^2}{2}$ is a volatility penalty.

$\therefore$ only want stock whose price drifts up $\smile$

so you want $(\mu - \frac{\sigma^2}{2})$ to be +ve

Option Prices and Geom Brown Mo.

$$e^{-rT} E[(S(T) - k)^+] \rightarrow x^+ = \max\{0, x\}$$

$\rightarrow$ "you could have put your money elsewhere" penalty

k : pre-agreed strike price

T : ^{EXPIRY} ~~penalty~~ date

r : "risk-free" interest rate

$\rightarrow$ Solve this to win a Nobel Prize $\smile$

Bivariate Normal $k=2$

Generate $\mathbf{X} = \boldsymbol{\mu} + \mathbf{C}\mathbf{Z}$
vectors.

$$\mathbf{C} = \begin{bmatrix} \sqrt{\sigma_{11}} & 0 \\ \frac{\sigma_{12}}{\sqrt{\sigma_{11}}} & \sqrt{\sigma_{22} - \frac{\sigma_{12}^2}{\sigma_{11}}} \end{bmatrix}$$

$$\text{and } X_1 = \mu_1 + \sqrt{\sigma_{11}} Z_1$$

$$X_2 = \mu_2 + \frac{\sigma_{12}}{\sqrt{\sigma_{11}}} Z_1$$

$$+ \sqrt{\sigma_{22} - \frac{\sigma_{12}^2}{\sigma_{11}}} Z_2$$

→ This is
nothing more than a
lower Δ^2 matrix!
vals don't matter....

M8:1 Input Analysis

Input data can GLGO the simulation. So have to be careful!

Need a good set of RV's for things like arrival times.

- So:
- ① Collect Data
 - ② Pick distribution
 - ③ Perform GoF tests to see if ② $\xrightarrow{\text{fits}}$ ① OK

PLAN

Graph things (input data) with a histogram.

If you take enough samples/observations, histogram will converge to True dist. GLIVENKO-CANTELLI!

→ more observations the better. BUT too many can confound Choice!

Which Distribution?

- Eg.: Tri (min, max, "most likely")
- Unif → not much known, expect min and max "
 - Normal → Heights, weights, IQ's
 - Beta → Bounded data
 - Γ , Weibull, Gumbel, lognormal → RELIABILITY data!

→ You can also bail and use the "Empirical" / Sample dist

Unbiased Point Estimation

→ $\bar{x}$ and s^2 are statistics

→ μ and σ^2 are parameters

Statistics $\xrightarrow{\text{Estim}}$ parameters. you know this shit...

→ You have to show that a statistic is an Unb. Est. of a parameter.

i.e., $T[x]$ is unbiased for θ

$$\text{iff } E[T[x]] = \theta$$

⇒ For Exp(λ), $\bar{x} = \frac{1}{\lambda}$ **BUT** $\lambda \neq \frac{1}{\bar{x}}$

⇒ Normal, $E[s^2] = \sigma^2$

You can show these mathematically.

⇒ s^2 unbiased $\rightarrow \sigma^2$

BUT s BIASED $\rightarrow \sigma$ (!!!!)

What are good props of $T[x]$?

- ① Should be unbiased.
 - ② Low bias
 - ③ Low var
- $\downarrow B \uparrow V \Rightarrow$ Noisy, bad
 $\uparrow B \downarrow V \Rightarrow$ OVER CONFIDENCE

Note: $\text{Bias}(T) = E[T] - \theta$

~~What~~ diff. bet. what T is trying to est. and the actual param!

$$\text{MSE} \equiv E[(T - \theta)^2]$$

Mean Squared Error

$$MSE \equiv E[(T - \theta)^2]$$

after math....

$$MSE(T) \equiv \text{Var}(T) + \underbrace{(E[T] - \theta)^2}_{\text{Bias!}}$$

→ Lower MSE is better even if there is little Bias!

$$\text{Relative Eff. of } T_2 \text{ to } T_1 \equiv \frac{MSE(T_1)}{MSE(T_2)}$$

if $RE < 1$, pick T_1

→ if T is unbiased, then
 $MSE = \text{Var}(T)$ no bias term!

How to do Point Estimate? ???

Most popular is **MLE!**

↳ has some bias but OK on avg.

MLE: multiply all the pdf's and minimize.

$X_1, X_2, \dots, X_n \stackrel{\text{iid}}{\sim}$ Some dist w pdf $f(x)$

$$L[\theta] \equiv \prod^n f(x_i)$$

Each can have its own pdf!!

So, MLE of θ is value that $\text{Max}(L[\theta])$
(use derivative and then second derivative test!)

function of $(x_1, \dots, x_n)$ and is RV!
more obs and MLE changes!

Eg: $X_1, \dots, X_n \stackrel{\text{iid}}{\sim} \text{Exp}(\lambda)$

find MLE for λ

$$L(\lambda) \equiv \prod^n f(x_i) = \prod^n \lambda e^{-\lambda x_i} \\ = \lambda^n \exp\left(-\lambda \sum^n x_i\right)$$

Terrible math... BUT! $\ln()$ is one-to-one, so use that before maxing (taking derivative)

$$\Rightarrow \ln[L(\lambda)] = n \ln(\lambda) - \lambda \sum^n x_i$$

$$\Rightarrow \frac{d}{d\lambda} \left(\downarrow \right) = \frac{n}{\lambda} - \sum^n x_i = 0$$

$$\Rightarrow \boxed{\hat{\lambda} = \frac{1}{\bar{x}}} \quad \text{NOTICE THE HAT}$$

$$\hat{\lambda} = 1/\bar{x} \quad \text{OK}$$

IIIrd for $\text{Bern}(p)$,

$$\boxed{\hat{p} = \bar{x}}$$

$$\lambda \neq 1/\bar{x} \quad \text{NOT OK}$$

There were baby examples

Eg: $X_1, \dots, X_n \stackrel{\text{iid}}{\sim} N(\mu, \sigma^2)$

need to find simultaneous MLEs for μ and σ^2 ! $L(\mu, \sigma^2)$ is HUGE!

use same loge trick... **BUT** take partial derivatives

then,

$$\boxed{\begin{aligned} \hat{\mu} &= \bar{x} \\ \hat{\sigma}^2 &= \frac{\sum (x_i - \bar{x})^2}{n} \end{aligned}}$$

M8:2 | MLE has some bias!!

Eg: $s^2 \equiv \frac{\sum (x_i - \bar{x})^2}{(n-1)}$
unbiased

and $\hat{\sigma}^2 = \frac{\sum (x_i - \bar{x})^2}{n}$

biased...
(can prove)

So, $s^2 = \frac{n}{(n-1)} \hat{\sigma}^2$

So, $\hat{\sigma}^2$ may be biased but it has less variance!

→ $s^2 \approx \hat{\sigma}^2$ for large n so... whatever

TRICKY: Gamma (λ, r) ~ $X_1, \dots, X_n$

Do the same w/ $\ln(L(\psi, \lambda))$
trick | Find $L(r, \lambda)$

then, $\hat{\lambda} = \hat{r} / \bar{x}$

→ But $\hat{r}$ is tricky and needs the Digamma func: $\Psi(r) \equiv \frac{\Gamma'(r)}{\Gamma(r)}$

→ Now you need some way (bisection, Newton's) to find r that solves:

$n \ln(\frac{r}{\bar{x}}) - n \Psi(r) + \ln(\prod x_i) = 0$

uni: $\Gamma'(r) = \frac{\Gamma(r+h) - \Gamma(r)}{h}$

just good old Calculus... ☺

Eg: UNIF: $X_1, \dots, X_n \stackrel{iid}{\sim} \text{Unif}(0, \theta)$

Find MLE of θ

pdf $f(x) = \frac{1}{\theta}$ $0 < x < \theta$ LIMITS! CARE!!

$\Rightarrow L(\theta) = \begin{cases} \frac{1}{\theta^n} & \text{if } 0 \leq x_i \leq \theta \forall i \\ 0 & \text{otherwise} \end{cases}$

so $L(\theta)$ is maxed out @ $\frac{1}{\theta^n}$ i.e. at the smallest θ

so $\hat{\theta} = \text{Max}[x_i]$ ☺

Cannot be zero! if $L > 0$

NOTE!!! if asking $U(0,0)$ (paradox), it's the min value!!

THE INVARIANCE PROP of MLE

You've used this already!

→ if you have MLE of θ , $\hat{\theta}$ (and $f(\cdot)$ is some 1-to-1 function, $f(\hat{\theta})$ is the MLE of $f(\theta)$ ☺

⇒ Only for MLE! NOT for unbiased

$E[s^2] = \sigma^2$ BUT $E[\sqrt{s^2}] \neq \sigma$

No! if you have $\hat{\sigma}^2 = \frac{1}{n} \sum (x_i - \bar{x})^2$ $N(\mu, \sigma^2)$ then, because MLE and not unbiased, then, $\hat{\sigma} = \sqrt{\hat{\sigma}^2}$

Eg: $X_1, \dots, X_n \stackrel{iid}{\sim} \text{Exp}(\lambda)$ then $\hat{\lambda} = \frac{1}{\bar{x}}$ $P(X > x)$

Survival func $\bar{F}(x) \equiv 1 - F(x)$ $\xrightarrow{\text{CDF}} e^{-\lambda x}$
 $\Rightarrow \hat{\bar{F}}(x) = e^{-\hat{\lambda}x} = e^{-x/\bar{x}}$ ACTUAL SCIENCE

GOODNESS OF FIT

Have I picked an appropriate dist that models Reality?

So, test hypothesis

$$H_0: X_1, \dots, X_n \stackrel{iid}{\sim} \text{pmf/pdf } f(x)$$

$$\alpha \equiv P(\text{Reject } H_0 | H_0 \text{ is TRUE})$$

→ Type I Error!

There are many GoF's lent

- χ^2 GoF
- Kolmogorov-Smirnov GoF
- Anderson-Darling GoF
- Shapiro-Wilk (for $N(\mu, \sigma^2)$)

① χ^2 GoF test!

Idea is very simple. You partition the domain of $f(x)$ into some number. Then you compare how many ^{observed} are in the partitions, versus how many you expect in these partitions/bins.

The Statistic is

$$\chi^2 = \sum_{i=1}^k \left[\frac{(O_i - E_i)^2}{E_i} \right]$$

$$\text{if } \chi^2_0 \leq \chi^2_{\alpha, (k-1-s)}$$

then fail to reject H_0

Bin/partitions
of params to estimate

Accept...
grudgingly

EXAMPLES

~~Eg~~ YOU NEED
EQUI-PROB
PARTITIONS!
Makes life easy!

So for partitions k ,
you can get $a_i = \text{some}$
function of k .

Eg Uniform.

$$X_i \sim U(0, 1) \quad n=1000$$

divide into $k=5$ bins

Int	[0, 0.2]	(0.2, 0.4]	...
E_i	200	200	200...
O_i	172	187	1400

Then compute χ^2_0 . $v = k - 1 - s$
 $\downarrow$ $\downarrow$
 5 0

Eg Geometric

→ need p . But use MLE

$$\hat{p} = \frac{1}{\bar{x}} = \frac{70}{(1)(34) + 2(18) + 3(2) + 4(9) + 5(7)} = 0.476$$

→ need EXP. VALUE E_i !

use the Invar. prop.

$$E_x = n P(X=x) = n (1-\hat{p})^{x-1} \hat{p}$$

$f(x)$
 $P(X=x)$

No! ~~the holdup? Try?~~

IMPORTANT

Pick k, n
such that

- $E_i \geq 5$
(else combine!)
- n is at least 30

IF $k-1-s$ is too big,

$$\chi^2_{\alpha, v} \approx v \left[1 - \frac{2}{9v} + \frac{2}{9} \sqrt{\frac{2}{9v}} \right]^3$$

# defects	freq
1	34
2	18
3	2
4	9
5	7

Is this Geom?
???

MS: 3 | $E_x = \sum x \cdot p(x)$ (prob that you have n get from p Equi prob bin. defects) $n \ln p$ //

So make table
Problem! <
Remember: $E_x \geq 5$
So combine with the next row!

x	$P(X=x)$	E_x	O
1	0.4762	33.33	24
2	0.2494	17.46	18
3	0.1307		
4		4.79	9
≥ 5			

So now,
 $k = 4$
 $S = 1$ ($\because \hat{p}$ had to be estimated)

$\Rightarrow v = k - 1 - S = 2$

$$\chi^2_0 = \sum_{x=1}^k \left[\frac{(E_x - O_x)^2}{E_x} \right]$$

$\chi^2_{\alpha, v} = \chi^2_{0.05, 2} = 5.99$

$\chi^2_0 = 9.12$

$\chi^2_0 > \chi^2_{\alpha, v}$
 $\Rightarrow$ REJECT!
NOT GEOM!

BUT need λ . Use MLE $\hat{\lambda}$
 $\Rightarrow S=1$

So $\hat{a}_i = -\frac{1}{\hat{\lambda}} \ln(1 - \frac{i}{k}) = -\bar{x} \ln(\dots)$

$\hat{\lambda} = \frac{1}{\bar{x}}$

So if $n=100, k=5, \bar{x}=9.0$

$\hat{a}_i = (-9.0) \ln(1 - \frac{i}{5})$

So, make a table
 $(a_{i-1}, a_i], E_i, O_i$

$(0, 2.01]$	20	25
$(2.01, 4.60]$	20	27
$\vdots$	$\vdots$	$\vdots$
$k-1-S = (5-1-1) = 3$	100	100

Compute χ^2_0 and $\chi^2_{\alpha, v} (= \chi^2_{0.05, 3})$

Eg/ EXPONENTIAL

Continuous! Can use this Strategy for others. EQUIPROB BINS

If you have k bins, pick the intervals S X_i falls in bin A_i w/prob $(1/k)$ < keep it simple!

STRATEGY. CDF $F(a_i) = P(X \leq a_i)$

$\alpha, 1 - e^{-\lambda a_i} = \frac{i}{k} \quad i=1, 2, \dots, k$

$\alpha, a_i = -\frac{1}{\lambda} \ln(1 - \frac{i}{k})$

Eg/ Weibull

if $X_1, \dots, X_n \sim W(r, \lambda)$ Simple lookup!

Then, $\hat{a}_i = -\frac{1}{\hat{\lambda}} \ln(1 - \frac{i}{k})$

We need to estimate 2 params so $S=2$

But! Solving for $\hat{\lambda}$ and $\hat{r}$ is a nightmare.

Same strategy though...

$\hat{a}_i = \frac{1}{\hat{\lambda}} \left[-\ln(1 - \frac{i}{k}) \right]^{1/r} \quad i=1, 2, \dots, k$

$\hat{a}_i$ is hard....

How to get $\hat{\lambda}$ or $\hat{\tau}$?
 use Newton's method!
 It converges pretty quickly.

then,

$$\hat{\lambda} = \left(\sum_i x_i \hat{\tau} \right)^{-1/\hat{\tau}}$$

and, $\hat{a}_i = \frac{1}{\hat{\lambda}} \left[-\ln \left(1 - \frac{i}{n} \right) \right]^{1/\hat{\tau}}$
 ($\therefore$ invariance prop.)

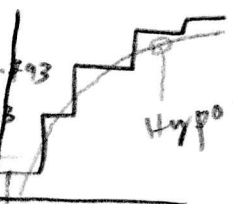
You have to look up $D_{\alpha, n}$
 Remember "histogram will converge to $F(x)$ for large n "?
 That's Glivenko - Cantelli.
 Applies here. Question: Does it?
 What's the deviation D ?

Baby Example: $H_0: X_1, \dots, X_n \stackrel{iid}{\sim} U(0, 1)$

$$D \equiv \max_{x \in \mathbb{R}} |F(x) - \hat{F}_n(x)| = \max_{0 \leq x \leq 1} |x - \hat{F}_n(x)|$$

Eg: MAKE ORDERED STATS $X_{(i)}$ CDF of Unif.
 $n=5$

X_i	0.039	0.706	0.016	0.198	0.793
$X_{(i)}$	0.016	0.039	0.198	0.706	0.793
i/n	0.2	0.4	0.6	0.8	1
$(i-1)/n$	0	0.2	0.4	0.6	0.8
FOO	0.184	0.361	0.402	0.094	0.207
BAR	0.016	-ve	-ve	0.106	-ve



Empirical CDF is step func!

$$FOO = \frac{i}{n} - X_{(i)}$$

$$BAR = X_{(i)} - \frac{i-1}{n}$$

So, $D^+ = 0.402$
 $D^- = 0.106$
 $\Rightarrow D = \max(D^+, D^-) = 0.402$

$$D_{\alpha, n} = D_{0.05, 5} = 0.565 \Rightarrow \text{ACCEPT!}$$

need table!
 specific to distro...

② KOLMOGOROV - SMIRNOFF Gof

- Works esp. well in "low data" situations. Good for others as well.
- Very Conservative; need lots of bad news to reject H_0

$$H_0: X_1, \dots, X_n \stackrel{iid}{\sim} \text{cdf of } F(x)$$

Empirical CDF $\hat{F}_n(x) \equiv \frac{(\# \text{ of } X_i) \leq x}{n}$

Hypothesized CDF: Pick something.

k-S measures the deviation of Emp. from Hypo!

Does the emp. CDF support H_0 ?
 REJECT if

$$D \equiv \max_{x \in \mathbb{R}} |F(x) - \hat{F}_n(x)| > D_{\alpha, n}$$

③ Anderson - Darling

④ Shapiro - Wilk (Normal Dist!)

⑤ Q-Q plots! (quick and dirty!)

"EMPIRICAL" means CDF from SAMPLE!
 can FIT a line through this!!

M8:4 Problem Children.

When to worry?

- 1) Little (or zero) data
- 2) Doesn't fit distros... "
- 3) Non-Stationary
- 4) Multi-variate / correlated "

① Little to no data

- Talk to Experts if no data
 - Min, Max, Mode?
 - Quantiles? ("95% falls into...")
 - Nature of Obs
 - Little Data? Try a good guess
 - Aves? Sums? Maybe $N(\mu, \sigma^2)$
 - Bounded? Maybe beta
 - Success/Fail? Maybe Bin / Bern
 - Reliability? Gamma, Weibull, Log Normal
 - Poisson?
- (Eg: Particles after fallout, Eg stocks!)

② Does not fit Distro "

- Maybe mix distros?
 - Smooth Empirical / Sample distro and bootstrap?
- "Sample from Sample itself..."

③ Non-Stationary "

- Non-Homo Poisson maybe?
- Use ARENA: "piece-wise constant rate function"

④ Multivariate "

- (Eg: Height / Weight)
- Use multivariate normal?
 - (Estimating covariances is not Easy)

Serially Correlated

$$X_1 \xleftarrow[\text{by}]{\text{affected}} X_2 \xleftarrow[\text{by}]{\text{affect}} \dots$$

Eg: unemployment rate, web traffic, fatigue.

- Use time-series models like ARMA(p,q), ARTOP, EAR(1)
- JUST BOOTSTRAP FROM EMPIRICAL BROOD.....

ARENA SHIT

Has an "Input Analyzer"

SMALL p-value? REJECT H_0
 < 0.05 ? $\uparrow$
 $< \alpha$? $\uparrow$ $F \neq 5!$

MQ:1 OUTPUT ANALY

Dave's Favorite ♡

What about the output?

It's never iid IT'S A LIE!

① Finite Horizon Analysis ^{T6PIX}
 ↓ (terminating) duplications, etc
 (simulation initialization)

② Steady state analysis ^{long term}

∴ Input is RVs → Output is estimates
 you only get ESTIMATES!

- Means are OK but for the full story,
 - Need Variances!
 - Need Quantiles
 - Need Probabilities
- } in addition

① Point Estimators
 ② Confidence Intervals
 are how you analyze output.

- Tough to apply "Classical" state methods.

Classical Stats WARNINGS

Assume $Y_1, \dots, Y_n$ are identically dist but not IND!

mean? $E[\bar{Y}_n] = \frac{1}{n} \sum E[Y_i] = \mu$

no mean is always unbiased! ♡

Variance? $Var[\bar{Y}_n] = Cov(\bar{Y}_n, \bar{Y}_n)$

$$= \frac{1}{n^2} \sum_{i=1}^n \sum_{j=1}^n Cov(Y_i, Y_j)$$

if $R_k \equiv Cov(Y_i, Y_{i+k}) \forall k=0,1,2$

$$V[\bar{Y}_n] = \frac{1}{n} \left[R_0 + 2 \sum_{k=1}^{n-1} \left(1 - \frac{k}{n}\right) R_k \right]$$

VERY FAMOUS EQUATION

⇒ It relates the variance of the sample mean to the covariance of the process!

So, $\sigma_n^2 \equiv n V[\bar{Y}_n] = R_0 + 2 \sum_{k=1}^{n-1} \left(1 - \frac{k}{n}\right) R_k$
 just a defn...

also define VARIANCE PARAM

$$\sigma^2 \equiv \lim_{n \rightarrow \infty} \sigma_n^2 = R_0 + 2 \sum_{k=1}^{\infty} R_k$$

Simplify.

$$\sigma^2 = \sum_{k=-\infty}^{\infty} R_k$$

$$\sigma^2 = R_0 + 2 \sum_{k=1}^{\infty} R_k$$

So! If Y_n are all iid,
 $R_k = 0 \Rightarrow \sigma^2 = \sigma_n^2 = R_0 = \text{Var}(Y_1)$
 Baby stat! $\downarrow$
 $\therefore$ all Y_n have same mean and variance!

$\rightarrow$ If dependent, all the covariance effects are added up! $\sigma^2 \propto \sigma_n^2$ for large n

BUT! $\sigma_n^2 \gg \text{Var}(Y_1)$!

This causes all sorts of probs with CI's!!

Note: $\frac{\sigma_n^2}{\sqrt{[Y_1]}}$ is "sort of" the # of obs you need to be

Equivalent to one INDEP obs.

So in example,

$$\sigma^2 = \frac{1+\phi}{1-\phi} \neq \text{Var}(Y_1) = 1$$

$\therefore N(0,1)$

If $\phi = 0.9$, you need 19 obs to get the info equiv. to one indep. obs!

Eg) 1st order autoregressive!

$$Y_i = \phi Y_{i-1} + \epsilon_i \quad \forall i \in \mathbb{N}$$

$-1 < \phi < 1$ $\sim N(0,1)$ $\sim N(0, 1-\phi^2)$
 $\sim N(0,1)$ $\underbrace{Y_0 \sim N(0,1)}_{\text{start with}} \underbrace{\text{ind of } Y_0}_{\text{ind of } Y_0}$

Here, $R_k = \phi^{|k|} \quad \forall k$

Eg: "Corr. bet. this month's unemployment and the unemployment k months from now" $\smile$

So! $\sigma^2 = \sum_{k=-\infty}^{\infty} \phi^{|k|} = \frac{1+\phi}{1-\phi}$

Sample Variance?

$$S_Y^2 = \left(\frac{1}{n-1}\right) \sum (Y_i - \bar{Y}_n)^2$$

$\rightarrow$ IID: S_Y^2 is unbiased for $R_0 = \text{Var}(Y_1)$

also unbiased for σ^2 $\leftarrow \sigma_n^2$

$\rightarrow$ Dependent: Stuck....

Not such a great estim.

Assume: Y_i are iid

mean μ , cov. func R_k

$$E[S_Y^2] = R_0 - \frac{2}{n-1} \sum_{k=1}^{n-1} \left(1 - \frac{k}{n}\right) R_k$$

If large enough, CLT kicks in and says that Each Z_i is $\approx N(0, \text{Var}(Z_i))$

$$S_z^2 \approx \frac{\text{Var}(Z_i) \chi^2(r-1)}{(r-1)}$$

Exp. Var. Variance

and, importantly,

$$\mu \in \bar{Z} \pm t_{\frac{\alpha}{2}, r-1} \sqrt{\frac{S_z^2}{r}}$$

IMPORTANT

sample mean!

sample variance

Confidence Interval

$\mu = \text{Exp. Val of Sample Means } E[\bar{Y}_m]$

How to shrink CI? ie,

"How many more replications are required?"

$$\text{let } H \equiv t_{\frac{\alpha}{2}, r-1} \sqrt{\frac{S_z^2}{r}}$$

be the half length of CI

we want some

$$\epsilon \equiv t_{\frac{\alpha}{2}, r^*-1} \sqrt{\frac{S_z^2}{r^*}}$$

desired $r^* > r$ is number of more reps!

→ NO GUARANTEE tho!

With some math,

$$r^* = \left(\frac{H}{\epsilon}\right)^2 r$$

now you need to recalculate all of mean, vars, CI!

This implies that if you want to eg: reduce the CI by ^{factor of} 10, you need 100x more replications

Quantile Shift

Eg: Max Wait time during r reps. Say this is W_i

put things in order

$$W_{(1)} \leq W_{(2)} \dots W_{(r)}$$

percentile 0.9 = 90th etc

then, if quantile is ξ_p , the point estimator is

$$\hat{\xi}_p \equiv W_{(\lfloor rp + 0.5 \rfloor)}$$

cont. corr.

CI for this is hard to derive. It is bounded by j and k .

$$j = \lfloor rp + 0.5 - Z_{\frac{\alpha}{2}} \sqrt{rp(1-p)} \rfloor$$

$$k = \lceil rp + 0.5 + Z_{\frac{\alpha}{2}} \sqrt{rp(1-p)} \rceil$$

Eg: $\xi_{0.9}$, $r = 1,000 \Rightarrow \hat{\xi}_p = W_{(900)}$

and 95% CI = $[W_{(881)}, W_{(920)}]$

M9:2 ~~No~~ we have:

$$E[S_Y^2] < \text{Var}(Y_i) \ll nV[Y]$$

$\therefore$ all those R_k functions produce output that add up!

$\Rightarrow$ Do not use $\frac{S_Y^2}{n} \xrightarrow{\text{to estimate!}} \text{Var}(\bar{Y}_n)$

if iid $100(1-\alpha)\%$ CI

$$\mu \in \bar{Y}_n \pm t_{\frac{\alpha}{2}, n-1} \sqrt{\frac{S_Y^2}{n}} \quad \text{iid}$$

BUT in the dependent case, the true CI will have coverage $\ll (1-\alpha)\%$ CAREFUL!

FINITE HORIZON SIMS

- $\rightarrow$ Discrete or Cont.
- $\rightarrow$ Use Replication!
- $\rightarrow [0, T] \rightarrow$ known or Random!

So from output samples,

$\bar{Y}_m$ is unbiased estimate of 0

$\rightarrow$ mean is ok

BUT cannot use variance!
 $V(\bar{Y}_m) \neq \frac{\text{Var}(Y_i)}{m}$

To fix this problem with variances, do Independent Replications

- $\rightarrow$ Do r reps for m obs.
- $\rightarrow$ Reinit each with a different seed Random!

so (r) reps for (m) obs

$$Z_i = \frac{1}{m} \sum_{j=1}^m Y_{i,j}$$

~~Now take the mean of Z_i~~

"Customer j 's waiting time from run/replication i "

$\therefore$ Each run is started from the same conditions, they are iid!

Now the GRAND MEAN!

$$\bar{Z}_r = \frac{1}{r} \sum Z_i$$

$$\text{and } S_z^2 = \left(\frac{1}{r-1}\right) \left[\sum (Z_i - \bar{Z}_r)^2\right]$$

$$\Rightarrow \frac{S_z^2}{r} \text{ is a reasonable estimate of } \text{Var}(\bar{Z}_r)$$

M9:2 | What about Sims?

- Three ways:
- ① CI's
 - ② Var. Reduction
 - ③ Rank & Select

① CI's

Run replication w/ diff ~~prng~~ #s
 Compare ~~prng~~ using APPROX CI
 the same baby stats.

Z_{ij} replication i ... b
 datapoint from repl.

you can also use PAIRED CI as well.

② Var. Reduction

Common Random #s
 Use the same PRNGS to make decisions about system

Eg: A has serv. time θ_A
 B has " " θ_B

estimate $\hat{\theta}_A \rightarrow \theta_A$

$\hat{\theta}_B \rightarrow \theta_B$

good if $\hat{\theta}_A < \hat{\theta}_B$ (maybe)

BUT $\text{Var}(\hat{\theta}_A - \hat{\theta}_B) = \text{Var}(\hat{\theta}_A) + \text{Var}(\hat{\theta}_B)$

so we lose conviction ...

Using CRNs,

$$\text{Var}(\hat{\theta}_A - \hat{\theta}_B) = \text{Var}(\hat{\theta}_A) + \text{Var}(\hat{\theta}_B) - 2\text{Cov}(\hat{\theta}_A, \hat{\theta}_B)$$

→ Adds a lot of this! so you can drive down $\text{Var}(\hat{\theta}_A - \hat{\theta}_B)$

→ TIGHTER CI's!

→ Intentional

Intro of Correlation!!

System

Antithetic PRNGs

worse than system

→ This time, introduce

NEGATIVE correlation!!

→ Opposite day!

$$\text{So, } V\left(\frac{\hat{\theta}_A + \hat{\theta}_B}{2}\right) = \frac{1}{4} \left[V(\hat{\theta}_A) + V(\hat{\theta}_B) + 2\text{Cov}(\hat{\theta}_A, \hat{\theta}_B) \right]$$

$$\Rightarrow V\left(\frac{\hat{\theta}_A + \hat{\theta}_B}{2}\right) = \frac{1}{2} \left[V(\hat{\theta}_A) + \text{Cov} \right] < \frac{\text{Var}(\hat{\theta}_A)}{2}$$

The whole

goal with these two

methods in Var. Reduction

iid you can add

This is MLE! runs somehow! Eg. Monte Carlo

Eg: Use $U(0,1)$: 0.17 0.23 0.46 0.76
 (Run $\hat{\theta}_A$ with this)

THEN, Run $\hat{\theta}_B$ with $(1-0.17) (1-0.23) \dots$

THEN Average $\left(\frac{\hat{\theta}_A + \hat{\theta}_B}{2}\right)$ FTW!

③ Rank and Select

Dave loves this!

→ Specify the Prob of picking the best system from its competitors.

→ Other techniques (CI, ANOVA) tell us "at least one is better than rest". Eh... not enough.

→ How to pick? — Which mean?
 { Smallest Variance?
 Combo...

- Greatest mean? NORM
- Highest prob? Binom
- Best Player? MULTINOMIAL

INDIFFERENCE ZONE APPROACH! ~~Rank and Select~~

Take: $k \sim N(\mu, \sigma^2)$ population

Vector of $\mu = (\mu_1, \dots, \mu_k)$
 $\sigma^2 = (\sigma_1^2, \dots, \sigma_k^2)$

Order this:

$$\mu_{[k]} = \mu_{[1]} \dots \mu_{[k]}$$

you want the largest mean, you want UNKNOWN $\mu_{[k]}$ ☹

REQUIREMENT CS = "Correct Selection" ^{Correct Selection}

~~Choose~~ Choose δ^* and P^* (you do this)
 $\delta^* > 0$ $\frac{1}{k} < P^* < 1$

Smallest diff worth detecting!

So, $P(CS) \geq P^*$ when $\mu_{[k]} - \mu_{[k-1]} \geq \delta^*$
 even

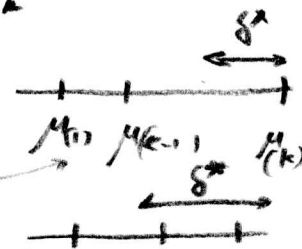
Prob of detection depends on

- $\begin{cases} n \\ \sigma^2 \\ \text{True means } \mu_k \end{cases}$

if $\mu_{[k]} - \mu_{[k-1]} \geq \delta^*$

you are in pref. zone

else you are in the indiff zone



If you pick $\mu_{[k-1]}$

no big deal $\therefore$

$\mu_{[k]} / \mu_{[k-1]}$ are so close.

YOU REALLY want to make the right select!
 - Could pick inferior system
 - You don't want to pick up your selection!!

Many ways! — Bechhofer (Single-stage)

Rinott (Two-stage)

Kim & Nelson (Sequential)

MQ: 3 Initializing Sims!

When to start? "Warm up"?

Queues cannot always be empty and idle (eg: Black Friday)

How to deal with "Init Bias"?

{ Visually, lot of statistical tests } this is on outputs!

So see the outputs and figure out where to truncate

too early? Bias. Too late? Waste!

Simplist Method: Welch's Method

- ① Just run a lot of reps
- ② Average them
- ③ Visually pick truncation point

② Steady-State Sims

Run some steady-state Sims, collect $Y_1, Y_2, \dots, Y_n$

then, $\bar{Y}_n = \mu$
good point of CS!

But we also need Variance $\hat{\sigma}^2$
 $\text{Var}(\bar{Y}_n)$

Instead use σ^2 , Variance Param

$$\sigma^2 \equiv \lim_{n \rightarrow \infty} n \text{Var}(\bar{Y}_n)$$

$$\sigma^2 = \lim_{n \rightarrow \infty} \left[R_0 + 2 \sum_{k=1}^{n-1} \left(1 - \frac{k}{n}\right) R_k \right]$$

iff R_k decrease quickly enough,

$$\sigma^2 = \sum_{k=-\infty}^{\infty} R_k$$



moving avg

Eg

for MA(1) process,

Cov. func.

$$Y_{i+1} = \theta E_i + E_{i+1} \quad N(0,1)$$

then, $\mu = [\bar{Y}_n], \sigma^2 = (1+\theta)^2$

Eg

$$\text{AR}(1): Y_{i+1} = \phi Y_i + E_{i+1}$$

then, $\mu = [\bar{Y}_n], \sigma^2 = \left(\frac{1+\phi}{1-\phi} \right)$

If you cannot get σ^2 this way, you can do Batch Means. Others exist!

Batch Means :

goal $\rightarrow$ Estimate σ^2
goal $\rightarrow$ Make CI for μ

$Y_1 \dots Y_n$ | Split b batches
of size m
 $\therefore n = bm$

$Y_1 \dots Y_m$ $Y_{m+1} \dots Y_{2m}$ $\dots$ $Y_{(b-1)m+1} \dots Y_{bm}$
 b_1 b_2 $\dots$ b_m
 $\downarrow$ mean $\downarrow$ mean $\downarrow$ mean
 $\bar{Y}_{i1}$ $\bar{Y}_{i2}$ $\dots$ $\bar{Y}_{bm}$

Formally, $\bar{Y}_{i,m} = \frac{1}{m} \sum_{j=1}^m Y_{(i-1)m+j}$

$\therefore$ CLT, each batch mean is $\approx$ Normal! So, Each mean

$\bar{Y}_{1,m} \dots \bar{Y}_{b,m} \stackrel{iid}{\approx} N\left(\mu, \frac{\sigma^2}{m}\right)$

What to do with all these means?
Estimate σ^2 of course!

$\hat{V}_B \equiv \left(\frac{m}{b-1}\right) \sum_{i=1}^b (\bar{Y}_{i,m} - \bar{Y}_n)^2$

The Batch Means Estimator
(χ^2)

It is a RV, so has E and V.
These are complicated... $\therefore$

$E[\hat{V}_B] = \sigma^2$

$V[\hat{V}_B] = \left(\frac{2\sigma^2}{b-1}\right)$

You can do MSE from there.
lots of math $\therefore$

What about CI's?

Symmetric!

$\mu \in \bar{Y}_n \pm t_{\frac{\alpha}{2}, b-1} \sqrt{\frac{\hat{V}_B}{n}}$

$\frac{S^2}{m} = \frac{\hat{V}_B}{n}$

$\Rightarrow$ also, S-S is 
and F-H is 
(bunch of smaller steps)

$\Rightarrow$ Recommendation is to take ~ 30 batches and then make the size m larger!

This makes ind and Normality problems go away as well...

Grand Mean!

M9:4 | Overlapping BM's

These are pretty remarkable!
The problem with BM is the correlation between batches ~

Simple $Y_1 \ Y_2 \ \dots \ Y_n \ Y_{n+1}$
 $\underbrace{Y_2 \ \dots \ Y_n}_{\text{overlap}}$

$$\bar{Y}_{i,m}^0 = \frac{1}{m} \sum_{j=i}^{i+m-1} Y_j$$

VERY correlated! But no prob!

$\hat{V}_0$ is the estimate for σ^2
Things get crazy...

$$\hat{V}_0 = \frac{m}{n-m+1} \sum_{i=1}^n (\bar{Y}_{i,m}^0 - \bar{Y}_n)^2$$

$$\frac{E[\hat{V}_0]}{E[\hat{V}_0]} \rightarrow 1, \quad \frac{V[\hat{V}_0]}{V[\hat{V}_0]} \rightarrow \frac{2}{3}$$

$\Rightarrow$ you are getting the same bias but less variance!

$\Rightarrow$ for large n , b batches,

$$\boxed{\text{dof} = \frac{3}{2}(b-1)} \quad \boxed{b = \frac{n}{m}}$$

$$\Rightarrow \underline{a\omega}, \quad \mu \in \bar{Y}_n \pm t_{\frac{\alpha}{2}, d} \sqrt{\frac{\hat{V}_0}{n}}$$

$\Rightarrow$ Use Overlapping BM's for large n !

Other Means

- ① Spectral Estimation
uses Freq. Domain instead of Time Domain!
- ② Regeneration
Sims "restart" at certain times $\rightarrow$ Regeneration Points.
You can break these into iid blocks.
BUT you might need to run the sim for long periods to know where to chunk into iid blocks.
- ③ Standardized Time Series
 - Standardize iid RVs into a single $N(\mu, \sigma^2)$ RV.
 - Brownian Bridges, process CLT, etc.

BATCHING makes shit NORM

M10:1) COMPARING

How to analyze a system and compare them?

① system: CI's from baby stats

CI's
|
mean
var
quantiles

② systems: ↓ paired system

③ system: Ranking & Selection

CI for μ of iid Normal Data

One Sample Want ¹⁰⁰ $(1-\alpha)\%$ CI

$$X_1, \dots, X_n \sim N(\mu, \sigma^2)$$

μ is to be estimated

σ^2 is unknown

$$\bar{X}_n \equiv \frac{1}{n} \sum X_i \sim N(\mu, \sigma^2/n)$$

and $S_x^2 \equiv \left(\frac{1}{n-1}\right) \sum (X_i - \bar{X}_n)^2$

<sub>Sam
Var</sub> $\sim \frac{\sigma^2 \chi^2_{(n-1)}}{(n-1)}$

Now say $T = \frac{\bar{X}_n - \mu}{\sqrt{S_x^2/n}}$

$$\sim \frac{N(0,1)}{\sqrt{\frac{\chi^2_{(n-1)}}{n-1}}} \sim \underline{t(n-1)}$$

So,

$$1-\alpha = P(-t_{\frac{\alpha}{2}, n-1} \leq T \leq +t_{\frac{\alpha}{2}, n-1})$$

this is how you can derive ...

$$\mu \in \bar{X}_n \pm t_{\frac{\alpha}{2}, n-1} \sqrt{\frac{S_x^2}{n}}$$

Diff bet. Two Means to compare two systems. ☺ Baby stuff!

Take: $X_1, \dots, X_n \stackrel{iid}{\sim} N(\mu_x, \sigma_x^2)$
 $Y_1, \dots, Y_m \stackrel{iid}{\sim} N(\mu_y, \sigma_y^2)$

Want: $\mu_x - \mu_y$ ← CI for this

Three ways: ① POOLED CI

Use when σ_x^2, σ_y^2 are equal, unknown

② APPROX CI: σ_x^2, σ_y^2 are unequal, unknown

③ PAIRED CI: when $\text{Cov}(X_i, Y_i) > 0$
{ you are collecting data in pairs from two competing pops!
so pairs are indep. but the datapoints in the pair are correlated! hence $\text{Cov}(X_i, Y_i) > 0$

NOTE:- If CI has 3 zero, you cannot say anything about $\mu_x - \mu_y \rightarrow$ Inconclusive

- $\mu_x - \mu_y$ left/Right of zero is common-sense.

① POOLED CI

When σ_x^2, σ_y^2 Equal but unknown
not too bad,

$$\mu_x - \mu_y \in \bar{X}_n - \bar{Y}_m \pm t_{\frac{\alpha}{2}, \nu} S_p \sqrt{\frac{1}{n} + \frac{1}{m}}$$

where $\nu = n + m - 2$ $\nu = n + m - 2$

S_p^2 is the pooled Var. Estimator

$$S_p^2 \equiv \frac{(n-1)S_x^2 + (m-1)S_y^2}{n+m-2}$$

just a linear combo!

② APPROX CI

When σ_x^2, σ_y^2 are unknown, Unequal

$$\mu_x - \mu_y \in \bar{X} - \bar{Y} \pm t_{\frac{\alpha}{2}, \nu} \sqrt{\frac{S_x^2}{n} + \frac{S_y^2}{m}}$$

where, $\nu \equiv \left(\frac{S_x^2}{n} + \frac{S_y^2}{m} \right)^2 - 2$

approx dof $\left(\frac{S_x^2/n}{n+1} + \frac{S_y^2/m}{m+1} \right)^2 - 2$

ROUND
THIS
DOWN!!!

③ PAIRED CI

Inter-pair is iid
Intra-pair is not.

→ This is nice because it
eliminates extraneous noise

Eg: Use same person to park
two different cars!

⇒ Use when possible! Prefer this

Take $X_1, \dots, X_n \stackrel{iid}{\sim} N(\mu_x, \sigma_x^2)$
 $Y_1, \dots, Y_n \stackrel{iid}{\sim} N(\mu_y, \sigma_y^2)$

and σ_x^2, σ_y^2 unknown, Unequal

so take Difference within pair,

$$D_i = X_i - Y_i \stackrel{iid}{\sim} N(\mu_D, \sigma_D^2)$$

BUT

$$\sigma_D^2 = \sigma_x^2 + \sigma_y^2 - 2\text{Cov}(X_i, Y_i)$$

BUT! We hope that this is big!

∴ it reduces σ_D^2 ☺

As usual, $\bar{D} \equiv \frac{1}{n} \sum D_i \stackrel{iid}{\sim} N(\mu_D, \frac{\sigma_D^2}{n})$

and $S_D^2 \equiv \frac{1}{n-1} \dots$

and $\mu_D \in \bar{D} \pm t_{\frac{\alpha}{2}, n-1} \sqrt{S_D^2/n}$

11.3
H.10

INDIFF: Bechhofer

(a) Single-stage
(NORM MEANS)

Make a decision at once

=> KNOWN COMMON VARIANCE!

k competitors, p^* and δ^*/σ

use these to look at
table to get (n)

mean
FOT

(can get w/sim also!

can get from multivar Norm
Frank

So this (n) from the table
tells you how many samples
to get from each of k competitors

Take the sample means μ_k

$$\textcircled{1} \mu_{[1]} = \mu_{[k-1]} = \mu_{[k]} - \delta^*$$

Slippage Conf: $\mu_{[k]}$

$\textcircled{2}$ is $\geq$ er than rest by
fixed amount

Least-Fav Conf:

> all other μ s in
pref zone are minimized!
 $P(c_s)$ is minimized!

now just get μ_k with
the largest value!

OBV, if δ^* is large
and p^* is small,
you need smaller (n) :)

(b) Pick w/highest Prob (or other
INDIFF param)

Goal: select Bern pop w/highest
success parameter.

B_{SH}

"Most of X, Best of Y, etc..."
whichever

This is spiritually the
same as (a), Bechhofer!

OK if independent!

k Bern populations with
 $p_1, \dots, p_k$, order $p_{(1)} \dots p_{(k)}$

take: (p^*, Δ^*) , $\frac{1}{k} < p^* < 1$
 $0 < \Delta^* < 1$

$P(\text{Correct Selection}) \geq p^*$

whenever $p_{[k]} - p_{[k-1]} \geq \Delta^*$

"When the diff is pretty big, do
not fuck up selecting $p_{[k]}$...
want to pick it with prob p^* "

Eg $k=4$ treatment, $\Delta^*=0.10$ $p^*=0.95$
table says $n=212$

pick $Y_{k,n}$ with most successes!

③ Multinomial

- "Best candidate?"
- "Most watched?"
- "Which config maximizes the throughput?"

→ Different than Bernoulli because they kinda compete w/ each other.

→ This, indeed, generalizes Bernoulli/Binomial!

This is kinda like binning...

k outcomes

p_i is prob of i^{th} cell/ category

n indep. replications

Y_i # of outcomes in category i

k -variate discrete Vector R.V

$$Y = (Y_1, Y_2, \dots, Y_k)$$

$$\text{if, } P\{Y_1 = y_1, \dots, Y_k = y_k\} =$$

$$\frac{n!}{\prod y_i!} \prod_{i=1}^k p_i^{y_i}$$

Again, if

$$P = \left(\frac{n!}{\prod_{i=1}^k y_i!} \right) \prod_{i=1}^k p_i^{y_i}$$

MUST > 0
add $+ + + \dots = 1$

this is a multinomial problem!

→ Think of that Harvard Card example.

eg Red, Blue, Green } Baby
 p_i $\frac{3}{6}$ $\frac{2}{6}$ $\frac{1}{6}$ } Example

toss 5 times, prob of 3Rs, 0Bs, 2Vs

$$P\{Y = (3, 0, 2)\} = \frac{5!}{3!0!2!} \left(\frac{3}{6}\right)^3 \left(\frac{2}{6}\right)^0 \left(\frac{1}{6}\right)^2$$

□

eg Same deal but you don't know the p_i !
"Which candidate do you like?"

Eg: R, B, G? "Select most prob. Color"

- Run Simulation (say 5 times)

- Use Eq. for $P = () \Pi$ to

Calculate prob of Red, B, G being picked

⇒ Randomize for ties!

M10
140:4 - So list outcomes favorable to R, B, G for Red:

Out	Contrib
(5,0,0)	(—)
(4,1,0)	(—)
!	
(3,0,2)	✗
(2,2,1)	(0.5) (—)
(2,1,2)	(0.5) (—)

Tie break!

$$P(CS) = 0.6042 \checkmark$$

⇒ The most prob. outcome might be pref. to the highest Exp. Val.!

eg) A → \$5 w/p=1
B = { \$0 0.99
 \$100 0.01

E[A] = \$5 } BUT!
E[B] = \$10 } P(Profit A > Profit B)
= 0.99

SELECTION PROB

$$X_j = (X_{1j}, X_{2j}, \dots, X_{kj})$$

Categories observation.

$X_{kj} = 1$ if person j picked Category k

probs $p_1, \dots, p_k$ are unknown and $P[k]$ is the "best" (most prob.)

GOAL: Pick Category with largest $P[k]$

NOTE: This is an extension for CATEGORICAL vars!

Same shit: BUT you are dividing for θ^* !

$$\frac{1}{K} < P^* < 1 \quad \text{but} \quad \frac{P[k]}{P[k-1]} \geq \theta^*$$

now look @ table

This is single-stage!

Calculate Ordered Sums

"Which candidate did you like?"

Procedure achieved

$P(CS) \geq P^*$ in least fav config.

$$P_{[1]} = P_{[k-1]} = \frac{1}{(\theta^* + k - 1)}, \quad P_{[k]} = \frac{\theta^*}{(\theta^* + k - 1)}$$

very hard to achieve this!

M.BEM: Beckhoff, Elmaghraby, More

Eg] $k=3$ kinds of colas.

Need $P(\text{correct selection})$

$$= P(CS) \geq 0.95$$

"ratio is at least 1.4" (0*)

table means $n = 186$

So, let's say

$$Y_{1,186} = 26 \quad Y_{2,186} = 110 \quad Y_{3,186} = 26$$

Then winner $\rightarrow$ is cola #2

- Can stop when cola #2 cannot win (CURTAILED)
- Can sequential: stop when 'clear' winner (sample one-at-a-time)

CONVOLUTION

Binom(n, p)

Ex (3, 0.4)

Sum of Bern(p)

$U = 0.75, 0.80, 0.33$

$$X = \underset{(\because \leq p)}{0} + \underset{(\because \leq p)}{0} + \underset{(\because \leq p)}{1} = 1$$

Triangula

Sum of 2 U's ($0, 1$)

Erlang(λ)

$X = -\frac{1}{\lambda} \prod U_i$ Sum of Exp's!

Normal

If sum U_i 's, CLT $\Rightarrow X \sim N\left(\frac{n}{2}, \frac{n}{12}\right)$

So $Y = \sum U_i - 6 = N(0, 1)$

AP

Always Works
DISC
CONT

Other CONT

$t(x)$: Majorize (always $> f(x)$)

$f(x)$: pdf

$$C = \int t(x) dx$$

$$g(x) = \frac{f(x)}{t(x)} = \frac{f(x)}{C h(x)}$$

loop until $U \leq g(x)$

Half Normal

$$f(x) = \frac{2}{\sqrt{2\pi}} e^{-x^2/2} \quad x \geq 0 \quad | \quad t(x) = \sqrt{\frac{2e}{\pi}} e^{-x}$$

$$h(x) = \frac{f(x)}{t(x)} = e^{-x}$$

$$C = \int_0^\infty t(x) dx = \sqrt{\frac{2e}{\pi}}$$

$$g(x) = e^{-(x-1)^2/2}$$

Unif(a, b)

Make U

$U > a$? Accept $X \leftarrow U$

else loop

$$Z = \begin{cases} -X & \text{if } U \leq 1/2 \\ X & \text{if } U > 1/2 \end{cases}$$

Poisson(λ)

Multiply U until smaller than $e^{-\lambda}$
see note for example.

OR $\rightarrow \lambda \geq 20 \Rightarrow \frac{X - \lambda}{\sqrt{\lambda}} \approx N(0, 1)$

Make $Z \sim N(0, 1)$ (any technique)

then $X = \text{Max}(0, \lfloor \lambda + \sqrt{\lambda} Z + 0.5 \rfloor)$ (center correction)

]-28360

$$2N(0,1)$$

$$2N(0,1) + N(0,1) \\ = N(0,2) + 3$$

1234321

0.005
0.995

$$X = \frac{1}{3} e^{3X} = 2V$$

$$3X = \ln(2V)$$

$$X = \frac{2}{3} \ln(V) - \frac{1}{3}$$

$$\text{Cov}(2X, -3Y + 2)$$

$$-6 \text{Cov}(X, Y)$$

P(

Composition

$$F(x) = \sum_{j=1}^{\infty} p_j \cdot F_j(x) \leftarrow \text{pick this appropriately.}$$

Laplace

split up, then
(see notes)

$$X \leftarrow \begin{cases} \ln(u) & \text{w.p. } 1/2 \\ -\ln(u) & \text{w.p. } 1/2 \end{cases}$$

OTHER

Box-Muller

Norm
 $N(0,1)$

$$\begin{aligned} Z_1 &= \sqrt{-2\ln(U_1)} \cos(2\pi U_2) \\ Z_2 &= \sqrt{-2\ln(U_1)} \sin(2\pi U_2) \\ &\text{RADS!!!} \end{aligned}$$

Polar

Norm
 $N(0,1)$

$$\begin{aligned} U_1, U_2 &\sim U(0,1) \\ V_i &= 2U_i - 1 \\ W &= V_1^2 + V_2^2 \end{aligned}$$

loop: $W > 1$?

regenerate/reject U_1, U_2

$$\text{else: } Y = \sqrt{\frac{-2\ln(W)}{W}}$$

$$\text{and } Z_i = V_i \cdot Y$$

Order stats

min?

make $\min\{X_1, \dots, X_n\}$ with just n U_i !

$$G(y) = 1 - [1 - F(y)]^n$$

$$\text{InvT: } Y = F^{-1}[1 - (1 - U)^{1/n}]$$

$$\text{Eg: } G(y) = 1 - [1 - (1 - e^{-\lambda y})]^n$$

$$\text{for Exp} = 1 - e^{-\lambda y^n} \sim \text{Exp}(n\lambda)$$

$$\Rightarrow Y = -\frac{1}{n\lambda} \ln(U) \quad \checkmark$$

Poisson

$$\text{Stream of times: } T_0 \leftarrow 0, T_i \leftarrow T_{i-1} - \frac{1}{\lambda} \ln(U)$$

Fixed times: n linear? Gen $U_n \rightarrow \text{Sort} \rightarrow \text{Done!}$

$$\text{bet } [a, b] \rightarrow T_i \leftarrow a + (b - a) U_{(i)}$$

$$2N(0,1) + 3 = N(3,4) \quad ???$$

$$N(0,1) < -1$$

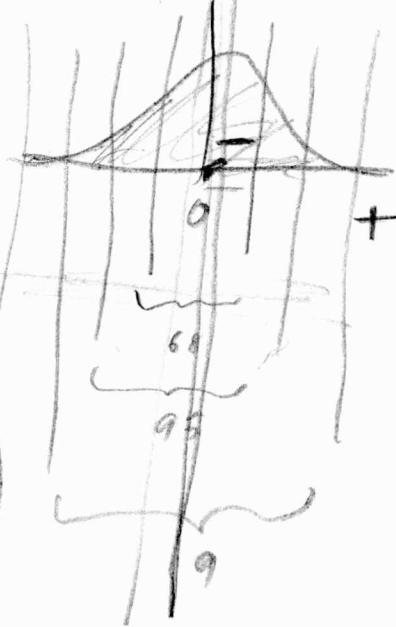
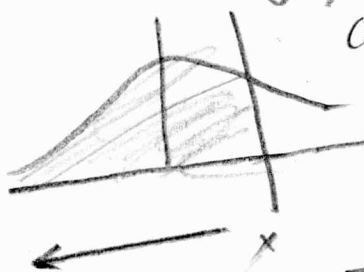
$$\Rightarrow N(0,1) < -\frac{1}{\sqrt{2}}$$

$$\text{Var}(aX + bY + cZ + dP)$$

Set $n=0$ to get U break / Split!

$$f(x) = \begin{cases} - & x \leq 0 \\ - & x > 0 \end{cases}$$

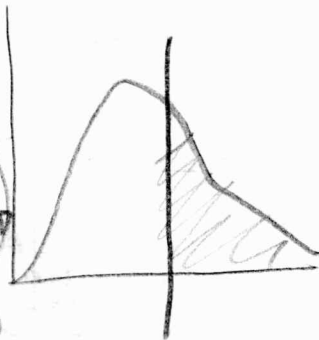
U? make $x=0$
Can see in graphs!!



$$P(N(\mu, \sigma^2) \geq 1.1)$$

$$P(Z \geq \frac{1.1 - \mu}{\sqrt{\sigma^2}})$$

$z = 2.57$ for $\alpha = 0.01$
REVERSE LOOKUP
2.57
0.995



TOGETHER

$$\left. \begin{aligned} X &\sim N(0,1) \\ Y &\sim N(0,1) \end{aligned} \right\} X+Y?$$

$$E[X+Y] = 1.0 + 1.0 = 0$$

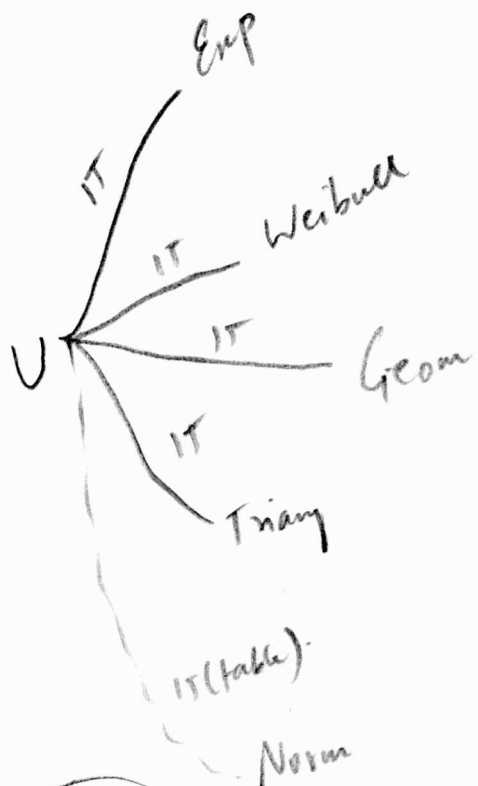
$$V[X+Y] = 1^2V[X] + 1^2V[Y] = 2$$

$$E[2X+3] = 2(0)+3 = 3$$

$$V[2X+3] =$$

Set, Product 1, 1, Prof order, Agent
grab from in order set attr.

I love you



find X_{39}
w/ cycle len (7)?
 $39 \div 7 = 5 \text{ remainder } 4 \Rightarrow X_4$

$$X > \sqrt{3}$$

$$\frac{X}{\sqrt{3}} > 1$$

$$0 \left(\frac{1}{\sqrt{3}} \right)^2 \cdot 1$$

$E[X - \sqrt{3}] = -\sqrt{3}$
ARENA
NORM(μ, σ^2)
STDEV not variance!

0.3 1 0.3
6.2 3 0.5
0.5 3 0

11011
168422'
 $16+8+0+2+1 = 27$
 $\frac{27}{32}$

Emp
no idea of dist!
→ see notes

18 mod 8 = 2 X_1
0 13 mod 8 = 5 X_2

+ - + + + + + + +
+ - + + + + + + +

inv trans

Exp $X = -\frac{1}{\lambda} \ln(u)$ or $-\frac{1}{\lambda} (1-u)$

Weib $X = -\frac{1}{\lambda} [-\ln(u)]^{1/\beta}$

Triangle $X = \sqrt{2u} \quad u < \frac{1}{2}$
 $X = 2 - \sqrt{2(1-u)} \quad u \geq \frac{1}{2}$

Bernoulli(p)

if $U_i \leq p \Rightarrow X=1$
else 0

Geom

→ $X = \left\lceil \frac{\ln(u)}{\ln(1-p)} \right\rceil = \left\lceil \frac{\ln(1-u)}{\ln(1-p)} \right\rceil$

→ Count Bern(p) trials
until success $p=0.5$
0.75 0.83 0.46 $\leq p \Rightarrow X=3$

norm

close form no !!
table !! $U=0.975 \Rightarrow Z = \Phi^{-1}(U)$
 $= 1.96$

Gen: $Z \sim N(3, 16)$ } $\mu + \sigma Z$
 $\Rightarrow X = 3 + 4Z$
 $\Rightarrow Z = 3 + 4\Phi^{-1}(U)$

Poisson

same as

other

compute f(x), add, slide!

$X \sim P(X=x) F(x)$
0 0.6 0.6 [0, 0.6]
1 0.3 0.9 [0.6, 0.9]
2 0.1 1.0 [0.9, 1]

OR (BCNA)
Crude approx

$Z = \Phi^{-1}(U) = U^{0.135}$
 $\sqrt{(1-U)^{0.135}}$
 $\forall 0.00134 \leq U \leq 0.98865$
 $\frac{0.1975}{0.1975}$

NORMAL $f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$ $F(x) = \int_{-\infty}^x f(x)$

$N(\mu, \sigma^2)$ $E[X] = \mu$, $V[X] = \sigma^2$, mgf = $M_X(t) = \exp\left(\mu t + \frac{1}{2}\sigma^2 t^2\right)$

- $X \sim N(\mu, \sigma^2) \Rightarrow aX+b \sim N(a\mu+b, a^2\sigma^2)$
- $X_1^2 + X_2^2 + \dots + X_n^2 \sim N\left(\sum_{i=1}^n a_i \mu_i + b_i, \sum_{i=1}^n a_i^2 \sigma_i^2\right)$
iid, N
- Standardize $Z = \frac{X-\mu}{\sigma} \sim N(0,1)$

N(0,1) $f(x) = \phi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$ $F(x) = \Phi(x) = \int_{-\infty}^x \phi(x)$

$P(Z \leq a) = \Phi(a)$ $P(Z \geq b) = 1 - \Phi(b)$ $P(b \leq Z \leq a) = \Phi(b) - \Phi(a)$

$\Phi(0) = 1/2$ $\Phi(b) = 1 - \Phi(-b)$ $P(b \leq Z \leq b) = 2\Phi(b) - 1$

CLT Any dist get $X_1, \dots, X_n$. The sample mean $\bar{X} = \frac{1}{n} \sum X_i$

(is $\sim N(\mu, \sigma^2/n)$) Variance goes down!

$n > 15$ Sample symmetric & unimodal $Z_n = \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \xrightarrow{d} N(0,1)$

Bern(p) $f(x) = \begin{cases} p & \text{if } x=1 \\ q & \text{if } x=0 \end{cases}$ $E[X] = p$ $V[X] = pq$ $mgf = pe^t + q$

Binom(n,p) "k successes in n trials" $E = np$ $mgf = (pe^t + q)^n$

$P(Y=k) = \binom{n}{k} p^k (1-p)^{n-k}$ $V = npq$

Eg: 100 trials, $P(Y \geq 84) = \sum_{i=84}^{100} \binom{100}{i} p^i (1-p)^{100-i}$

Geom(p) "n trials until 1st success" $E = 1/p$ $mgf = \frac{pe^t}{1-qe^t}$

$f(x) = (1-p)^{x-1} p$ $F(x) = 1 - (1-p)^x$

Eg: $P(Z > k) = q^k$, $P(Z \leq k) = 1 - (1-p)^k$

NegBin(r,p) "The rth success in n Bern(p) trials" $E = \frac{r}{p}$ $mgf = \left(\frac{pe^t}{1-qe^t}\right)^r$

$P(X=x) = f(x) = \binom{x-1}{r-1} q^{x-r} p^r$

Eg: FFFSFS $\Rightarrow NgB(3,p) = 7$

Poisson(λ) $f(x) = \frac{\lambda^x e^{-\lambda}}{x!}$ $E = \lambda$ $MGF = e^{\lambda(e^t-1)}$

$V = \lambda$

Animals on a time @ rate λ
 $\lambda = \frac{3}{hr} \Rightarrow \lambda = \frac{5}{30min}$

Unif(a,b) $f(x) = \begin{cases} \frac{1}{b-a} & a \leq x \leq b \\ 0 & \text{else} \end{cases}$ $F(x) = \begin{cases} 0 & x < a \\ \frac{x-a}{b-a} & a \leq x \leq b \\ 1 & x > b \end{cases}$ $E = \frac{a+b}{2}$ $V = \frac{(b-a)^2}{12}$

$mgf = \frac{e^{bt} - e^{at}}{t(b-a)}$ If $U \sim Unif(0,1)$, $E = \frac{1}{2}$ (duh)

Exp(λ) $f(x) = \lambda e^{-\lambda x}$ $F(x) = 1 - e^{-\lambda x}$ $MGF = \frac{\lambda}{\lambda - t}$ $t < \lambda$

$E = \frac{1}{\lambda}$ $V = \frac{1}{\lambda^2}$

Memoryless! Bulb made it 20 months. Will it make it 8 more? $\lambda = 1/20$

$P(X > 20+8 | X > 20) = P(X > 8) = 1 - [1 - e^{-\lambda(8)}] = e^{-\lambda(8)}$

Gamma(n, λ) $f(x) = \frac{\lambda^n x^{n-1} e^{-\lambda x}}{\Gamma(n)}$ $F_Y(y) = 1 - \sum_{j=0}^{n-1} \frac{(\lambda y)^j}{j!} e^{-\lambda y}$

$E = \frac{n}{\lambda}$ $V = \frac{n}{\lambda^2}$

$\Gamma(n) = \int_0^\infty t^{n-1} e^{-t} dt$ $\Gamma(1) = 1$ $\Gamma(n) = (n-1)! \Gamma(1)$

Tri(a,b,c) (Smallest, Most likely, Largest) (mean, mode, max)

$E = \frac{a+b+c}{3}$ V is a mess

RV modeling

BETA(a,b) $f(x) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} x^{a-1} (1-x)^{b-1}$ $E = \frac{a}{a+b}$ $V = \frac{ab}{(a+b)^2(a+b+1)}$

Can get Exp, Unif, N from this

Model RV's clamped to Interval (a,b)

RELATIONS Sum of Squared $N(0,1) \sim \chi^2(k)$ deg of freedom $E = k$ $V = 2k$

- $Z \sim N(0,1)$, $Y \sim \chi^2(k)$ then $T = \frac{Z}{\sqrt{Y/k}}$ is t-distribution! $\sim t(k)$ | $t(1) \sim \text{Cauchy}$
- $A \sim \chi^2(m)$ and $B \sim \chi^2(n) \Rightarrow \frac{A/m}{B/n} \sim F(m,n)$
- $\sum \text{Bern} = \text{Binom}$ $\sum \text{Geom} = \text{Neg Bin}$
- $\sum \text{Exp} = \text{Erlang}$ Eg: X, Y iid $\text{Exp}(\lambda)$ find $P(X+Y < 1)$
- $X_1, \dots, X_n \sim \text{Pois}(\lambda)$, $Y = \sum X_i \sim \text{Poisson}(\lambda)$
- $\sum \text{Geom}(p) = \text{Neg Bin}(n,p)$

CALC Derivative: $\cos(x) \rightarrow -\sin(x)$ $\frac{d}{dx}(f(x)g(x)) = f'(x)g(x) + f(x)g'(x)$

$f'(\arctan(x)) = \frac{1}{1+x^2}$ $\left[\frac{f}{g}\right]' = \frac{g f' - f g'}{g^2}$ $[f(g(x))]' = f'(g(x)) \cdot g'(x)$

Integrals $\int \frac{dx}{x} = \ln(x)$ $\int \cos(x) = \sin(x)$ $\int \frac{dx}{1+x^2} = \tan^{-1}(x)$ $\int f(x)g'(x) = f(x)g(x) - \int f'(x)g(x)$

HYPERGEOM Germ hut with replacement and learn variance!

a of type 1 $f(k) = \frac{\binom{a}{k} \binom{b}{n-k}}{\binom{a+b}{n}}$ $E = n \frac{a}{a+b}$

b of type 2 $P(X=k)$ $V = n \frac{a}{a+b} \left(1 - \frac{a}{a+b}\right) \frac{a+b-n}{a+b-1}$

k of type 1

$X \sim U(0,1) \Rightarrow 3X-1 \sim U(-1,2)$ $U_1, U_2 \sim \text{Triangular}(-1,0,1)$ Erlang 2, then $P(X \geq 2) = \sum_{i=0}^{\infty} \frac{\lambda^i e^{-\lambda}}{i!} = 0.4060$

POISSON: given rate function $\lambda(t) = t/2$ Exact by 3 arrivals before $t=2$? $N(s+t) - N(s) \sim \text{Poisson}(\int_s^{s+t} \lambda(u) du)$

So $N(2) = \text{Pois}(\int_0^2 t/2 dt) = \text{Pois}(1)$

So! $P(N(2) = 3) = P(\text{Pois}(1) = 3) = \frac{(1)^3 e^{-1}}{3!} = 0.0613$

If given $U_1 = 0.7, U_2 = 0.4$, then $\text{Erlang}_2(3) = -\frac{1}{3} \ln(U_1) = -\frac{1}{3} \ln(0.28) = 0.424$

POISSON: "Prob that time between arrivals" $\lambda = 3$ per hour $\Rightarrow k=2 \Rightarrow \text{Erlang}_2(\lambda)$

"Time between 5th and 6th" $\Rightarrow k=1$ so use $\text{Exp}(\lambda)$

"is > 1 hr?" $P(X > 1) = 1 - P(X \leq 1) = e^{-2} - 0.368$

Approx dist of $\text{Pois}(2) \sim N(\mu, \sigma^2) = N(\lambda, \lambda)$

$X \sim N(3,5)$ $F(X) \sim U(0,1)$

$Y \sim N(-2,4)$ $F^{-1}(U) \sim N(0,1)$

$X+Y: \mu = 3+(-2)=1$ $\sigma^2 = 5+4=9 \Rightarrow N(1,9)$

$X \sim N(3,4)$ $Y \sim (4,6)$

$E[2X-3Y] = 2(3) - 3(4) = -6$

$V[2X-3Y] = 2^2(4) + (-3)^2(6) = 70$

DECIDE 2 way by choice

n -way $\rightarrow$ choice

0.22 0.59 0.42 0.14 0.3465

Exact $\rightarrow 0.3413$

$(5(X_i+1)) \text{ mod } 6, X_0=3, U_2 = \frac{V_2}{6} = 0.11$

$5 \cdot 10 = 50, 5 \cdot 1, 5 \cdot 11, 5 \cdot 14, 5 \cdot 17, 5 \cdot 20, 5 \cdot 23, 5 \cdot 26, 5 \cdot 29, 5 \cdot 32, 5 \cdot 35, 5 \cdot 38, 5 \cdot 41, 5 \cdot 44, 5 \cdot 47, 5 \cdot 50$

$23-13=10$

$(X+Y < 2) \text{ Exp}(\lambda=1) \Rightarrow \text{Erlang}_2(1)$

Laplace(μ, b) $f(x) = \frac{1}{2b} \exp\left(-\frac{|x-\mu|}{b}\right)$

$F(x) = \begin{cases} \frac{1}{2} \exp\left(\frac{x-\mu}{b}\right) & \text{if } x \leq \mu \\ 1 - \frac{1}{2} \exp\left(-\frac{x-\mu}{b}\right) & \text{if } x \geq \mu \end{cases}$

$E = \mu$ ($= \text{mode} = \text{median}$) $V = 2b^2$ (Simple)

$MGF = \frac{e^{\mu t}}{1-b^2 t^2}$ $\forall |t| < 1/b$

RELATIONS

- If X is $\sim \text{Cauchy}$, $\bar{X}$ is $\sim \text{Cauchy}$
- Cauchy violates CLT!
- If $Z \sim N(0,1)$, $Y \sim \chi^2(n)$
- $\frac{Z}{\sqrt{Y/n}} \sim t(n)$ and $t(1)$ is Cauchy
- $\chi^2(2) \sim \text{Exp}(\frac{1}{2})$

ONLY IND IF ALL FOUR

$Z_1 = \tan(2\pi U_2)$

$\chi^2_2 = \sum_{i=1}^2 Z_i^2 = E(t_i) = F(1,1)$

$\chi^2(2) \sim \text{Exp}(\frac{1}{2})$

Process = Event + Activities
Process = Event + Activities
 Entity flow through modules/blocks network: **CREATE - PROCESS - DISPOSE**

Use a "Basic Process Template"

The PROCESS module

- Delay (self-service)
- Seize - Delay - Release (Queue formation)
- Seize - Delay (Eg. Hospital bed)
- Delay - Release

Process = Resource + Queue
 Name of Queue is "Process Queue"

$NQ(P1.Q) < NQ(P2.Q) \rightarrow 0 \text{ (false)}$
 $\rightarrow 1 \text{ (true)} \Rightarrow \text{Customer goes to Resource 1}$

Steps in Simulation Study: ① Problem Formulation ② Obj and planning ③ Model Building ④ Data Coll. ⑤ Coding (on a PC) ⑥ Verification (check CODE) ⑦ Validation (check MODEL or go back to 3,4) ⑧ Experimental Design ⑨ Run Exp. ⑩ Output Anal ⑪ Reporting

SYSTEM is a coll. of ENTITIES and is abstractly repr. by MODEL.
SYSTEM STATE: Vars and their values that desc system (math...)
 Eg: Single-Server Queue $q(t) = \# \text{ of people in } q \text{ at } t \text{ and } B(t) = 0 \text{ if no busy, } 1 \text{ if busy}$

EVENT is time when STATE changes unpredictably!
ACTIVITY is time of specified length $\rightarrow$ "Unconditional wait"
 Eg: Inter-arrival time. You run SIM with these. You run and then figure out the "Conditional Wait" from the SIM!

TIME: Moves forward, is a variable, not related to wall clock.

① Fixed Increment Time Adv: Update state @ nh , $h=0,1,2$
 Use for continuous shift or when only monthly (eg) data available.

② Next-Event Time Adv. (Clock tick @ next event, Update State and FEL. Every DISC-Event lang has an FEL)

Modeling approach: PROCESS-INTERACTION and EVENT-SCHEDULING
 ES is a bookkeeping nightmare. PI is better: focus on a generic customer and seq. of activ. and Ets flowing thru system.
 When Picking lang for SIM. ① \$\$\$ ② Ease of use ③ PTMES? ④ Feature

M3 $\frac{d}{dx} f(x) = f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$ (That's the definition, go...)
 So for small h , $f'(x) \approx \frac{f(x+h) - f(x)}{h}$ (use this shit.)

Eg: Solve $f'(x) = 2f(x)$
 $f(0) = 10$
 $h = 0.01$
 $\Rightarrow f(x+h) = (h)[2f(x)] + f(x) = (1+2h)f(x)$

EULER'S METHOD
 or Fixed Increment Time Approach
 what we want:
 $f(x+2h) = (1+2h)^2 f(x)$
 and $f(x+h) = (1+2h)f(x)$

MONTE CARLO INTEGRATION: Start with $I = \int_a^b g(x) dx$
 Now set $U_1, U_2, \dots \sim U(0,1)$
 Now set $I_i = (b-a)g[a+(b-a)U_i]$ $i=1,2,\dots,n$
 Then Sample Avg $\bar{I}_n$ is unbiased estimator! (Law of Large Ns)

$\bar{I}_n = \frac{1}{n} \sum_{i=1}^n I_i = \frac{(b-a)}{n} \sum_{i=1}^n g[a+(b-a)U_i]$

$\bar{I}_n \sim \text{Nor}(\bar{I}, \frac{\text{Var}(I_i)}{n})$
 Integrate anything with Uniform!!

100(1- α)% CI is $\bar{I}_n \pm Z_{\frac{\alpha}{2}} \sqrt{\frac{S_I^2}{n}}$
 where $S_I^2 = \frac{1}{(n-1)} \sum_{i=1}^n (I_i - \bar{I}_n)^2$
 Sample variance

PIE $\hat{\pi} = \frac{h}{\# \text{ in circle}}$ $\hat{\pi} \rightarrow \pi$ for large n !
 same for sphere. $\hat{\pi} \rightarrow \pi$ for large n !

SINGLE-SERVER
 Arrival time FIFO
 $A_i: \text{Max}(A_i, D_{i-1})$
 Service starts
 $W_i^q: \text{Waittime in queue}$
 $T_i = A_i$
 $W_i = D_i - A_i$
 time in system!
 $D_i = \text{Departure time} = T_i + S_i$

Avg. Waittime = $\frac{\sum W_i^q}{n}$
 Avg number in system = $\bar{L} = \frac{\sum (D_i - A_i)}{n}$
 Last depart time
 Eg: Avg # during "first 20 mins" = $\frac{\sum (20 - A_i)}{20}$

MAKE SHIT WITH Uni(0,1)
 Eg: Discrete Random? Ceiling $\lceil nU \rceil = X$. $[60] + 0.60$
 in two dice $0.5 \times \lceil nU \rceil + 0.5 \times \lfloor nU \rfloor$
 in 1, 5, 2, 5, 6

② if cannot be Eg: $P(X=x) = \begin{cases} 0.25 & \text{if } x=1 \\ 0.10 & \text{if } x=2 \\ 0.65 & \text{if } x=3 \\ 0 & \text{if } x=4 \end{cases}$
 then have Units again! So $U = 0.40 \Rightarrow x = 1.2$

③ Use to generate $\text{Exp}(\lambda)$!!
 if $X \sim \text{Exp}(\lambda)$, $F(x) \sim U(0,1)$
 $\Rightarrow X = F^{-1}(U) \Rightarrow 1 - e^{-\lambda x} = U$
 $\Rightarrow X = -\frac{1}{\lambda} \ln(1-U)$
 $\sim \text{Exp}(\lambda)$

④ Make PRNS $U_1, U_2, \dots$ Cycle length!
 Eg: $X_{i+1} = (5X_i + 1) \bmod(8)$
 and $U_{i+1} = \frac{X_{i+1}}{8}$
 to find X_{100} if $x_0 = 3$
 $100 \bmod 8 = 4 \Rightarrow$ we want x_4
 $x_0 = 3, x_1 = 4, x_2 = 7, x_3 = 6, x_4 = 5$

MGF's k^{th} moment of $X = E[X^k] = \int x^k f(x) dx$
 k^{th} central moment = $E[(x-\mu)^k]$ no $\text{Var}[X] = 2^{\text{nd}}$ central moment!
 $\text{mgf}(x) \equiv M_X(t) = E[e^{tx}]$ so that $E[X^k] = \frac{d^k}{dt^k} (M_X(t)) \Big|_{t=0}$
 now, if X has MGF, $M_X(t)$ then if Y is linear in $aX+b$,
 $M_Y(t) = e^{bt} M_X(at)$ Eg $X \sim \text{Exp}(\lambda) \Rightarrow Y = 3X+2$ then $M_Y(t) = e^{2t} \frac{\lambda}{\lambda - 3t}$

RIEMANN SUM for a, b, n
 $\int_a^b f(x) dx \approx (b-a) \sum_{i=1}^n f(a + \frac{b-a}{n} i)$

IFF X_i iid and $Y = \sum X_i$, $M_Y(t) = \prod (M_{X_i}(t))$ Can prove a lot of relations with this!

MEMORYLESSNESS: Geom(DISC) $\text{Exp}(\text{CONT})$ $P(Z > s+t | Z > s) = P(Z > t) = \frac{e^{-\lambda(s+t)}}{e^{-\lambda s}} = e^{-\lambda t}$
 $\Sigma \text{ Bern} = \text{Binom}$

CDF Non-decreasing
 $\lim_{x \rightarrow -\infty} F(x) = 0$
 $\lim_{x \rightarrow \infty} F(x) = 1$
 $P(X > x) = 1 - F(x)$
 $a < b \Rightarrow P(a \leq X \leq b) = F(b) - F(a)$

MARGINALS
 $f_X(x) = \sum_y f(x,y) \sim \int f(x,y) dy$
 $f_Y(y) = \sum_x f(x,y) \sim \int f(x,y) dx$
 CHECK LIMITS!!
 Using CDF (joint), $F(x,y)$
 $\rightarrow P(Y \leq y) = F_Y(y) = \lim_{x \rightarrow \infty} F(x,y)$

Failure Rate
 $s(t) = \frac{f(t)}{1-F(t)}$
 $s(t) = \frac{f(t)}{P(X > t)}$

SURVIVAL FUNC
 $\bar{F}(x) = 1 - F(x) = P(X > x)$

RELATIVE EFFICIENCY
 $\frac{\text{MSE}(T_1)}{\text{MSE}(T_2)}$
 if < 1 , want T_1

MLE
 $L(\theta) = \prod f(x_i; \theta)$
 for which this is max!
 use ln trick: $\ln L(\theta) \rightarrow$ then $\frac{d}{d\theta} \ln L(\theta) = 0$

Normal DIST ONLY
 CI when σ^2 known: $\bar{X} - Z_{\frac{\alpha}{2}} \sqrt{\frac{\sigma^2}{n}} \leq \mu \leq \bar{X} + \sqrt{\frac{\sigma^2}{n}}$
 for μ
 when σ^2 UNKNOWN: $\bar{X} - t_{\frac{\alpha}{2}, n-1} \sqrt{\frac{S^2}{n}} \leq \mu \leq \bar{X} + \sqrt{\frac{S^2}{n}}$
 for μ
 for σ^2 , $\frac{(n-1)S^2}{\chi^2_{\frac{\alpha}{2}, n-1}} \leq \sigma^2 \leq \frac{(n-1)S^2}{\chi^2_{1-\frac{\alpha}{2}, n-1}}$

DISC(F(x), x) → cumulative

COMPOSITION: $F(x) = \sum_{j=1}^n p_j F_j(x)$ → pick the appropriate

Laplace: $X \leftarrow \begin{cases} \ln(U) & \text{w.p. } 1/2 \\ -\ln(U) & \text{w.p. } 1/2 \end{cases}$

OTHER: Box-Muller: Normal: $Z_1 = \sqrt{-2\ln(U_1)} \cos(2\pi U_2)$
 $Z_1, Z_2 \sim N(0,1)$ $Z_2 = \sqrt{-2\ln(U_2)} \sin(2\pi U_2)$

TRIVIA: $Z_1^2 \sim \chi^2(1)$ | $Z_1^2 + Z_2^2 \sim \chi^2(2) \sim \text{Exp}(1/2)$ RADSS!!
 $Z_1/2 \sim N(0,1)$ | $Z_1/2 \sim \text{Cauchy} \sim t(1)$ | $(X/n) \sim F(n, n)$

Polar: Normal: $U, U_2 \rightarrow$ then $V_i = 2U_i - 1$
 faster then $W = V_1^2 + V_2^2$
 then loop: $W > 1$? REJECT REGENERATE
 $W \leq 1$, then $Y = \sqrt{\frac{-2\ln(W)}{W}}$
 and finally, $Z_i = V_i \cdot Y$

Order Stats: Use just one U_i to get $\min\{X_1, \dots, X_n\}$
 $\rightarrow G(y) = 1 - [1 - F(x)]^n$
 solve for $Y = G^{-1}(U)$
 $\Rightarrow Y = F^{-1}[1 - (1-U)^{1/n}]$ Eg: $X_1, \dots, X_n \sim \text{Exp}(\lambda)$ so get Min
 $G(y) = 1 - e^{-\lambda n y}$ → cdf of $\text{Exp}(n\lambda)$
 $\Rightarrow Y = -\frac{1}{n\lambda} \ln(U)$

Poisson: Arrivals
 ① Stream of Times @ λ
 $T_0 \leftarrow 0, T_i \leftarrow T_{i-1} + A_i$ or $T_i \leftarrow T_{i-1} - \frac{1}{\lambda} \ln(U)$ 11
 ② Fixed # of Arrivals between $[a, b]$
 $\rightarrow$ make $U_1, U_2, \dots, U_n$ → SORT $U_{(1)}, U_{(2)}, \dots, U_{(n)}$
 $\rightarrow$ then simply, $T_i \leftarrow a + (b-a)U_{(i)}$ 11

Non-Homo Poisson: Arrival rate λ is function $\lambda(t) = t^2$ (eg)
 Eg: Exactly 4 arrivals bet. 1 and 2
 $N(2) - N(1) \sim \text{Pois}(\int_1^2 \lambda(t) dt) \sim \text{Pois}(7/3)$
 so $P(X=4) = \frac{e^{-7/3} (7/3)^4}{4!}$

FROM OLD: • SEQUENCE spooled in "Adv. Transfer"
 $X(6) \sim \text{Erlay}_3(1/2)$ LEAVE $\leftrightarrow$ ENTER
 Do this by "Moving"
 generally,
 $X \sim \text{Erlay}(k, \lambda) \sim \text{Erlay}_k(\lambda)$
 $\Rightarrow 2\lambda X \sim \chi^2_{2k}$
 iid normal? MLE has invariant
 $X_1 = 3, X_2 = 2, X_3 = 7$ find MLE of $E[X_i^2]$
 $\Rightarrow E[X_i^2] = \hat{\mu}^2 + \hat{\sigma}^2 \rightarrow \frac{n-1}{n} S^2$
 $\Rightarrow \hat{\sigma}^2 = \frac{4^2 + 14}{3} = 20.67$
 • $P(W(a) < W(b)) = 1/2$ Brownian
 • Brownian Bridge $A = \int B(t) dt$
 $\text{find } P(P > \frac{1}{\sqrt{2}}) = P(N(0, 1/2) > \frac{1}{\sqrt{2}}) \Rightarrow P = P(N(0,1) > 1) = 0.1587$

• Use PRNS for $I = \int_0^1 e^x dx$. $\hat{I} = \frac{1}{n} \sum e^{U_i}$, $\hat{I}_n = \frac{1}{n} \sum e^{1-U_i}$
 $\Rightarrow \hat{I}_n = \frac{1}{2} (\frac{1}{2} + \frac{1}{2})$ antithetic!
 • If X, Y iid $\text{Exp}(1)$
 then $E[X^2 Y^2] = E[X^2] E[Y^2] = \int \Gamma \cdot \Gamma = 2 \cdot 2 = 4$
 $k=3$ $k=4$
 $\lambda=1$ $\lambda=1$
 • Have U 's? $n=14$ of them? Box-Muller aside...
 realization (approx Normal) = $\sum U_i - 14/2$
 • STANDARD BROWNIAN PROCESS $\sqrt{t}$
 dist of $W(0) + 2W(1) + 3W(2)$?
 $\rightarrow W(0) = 0$ so $Y = 2W(1) + 3W(2)$
 $E(Y) = 2E[W(1)] + 3E[W(2)] = 4V[W(1)] + 9V[W(2)] = 4(1) + 9(2) = 22$
 $V(Y) = 4V[W(1)] + 9V[W(2)] + 2(2)(3)\text{Cov}(1,2) = 4(1) + 9(2) + 12(1) = 34$

• σ^2 IS ALWAYS $\frac{S(x_i - \bar{x})^2}{n}$
 Sample variance low dof: $n-1$
 • ADJUSTING C.I.'s: Eg. CI in $[-10, 110]$ @ 90%
 $n=4$
 Adjust for 95%? $[-10, 110] = \bar{x} \pm t_{\alpha/2, 3} \cdot s$
 $\bar{x} = 50$, Halfwidth = $\frac{110 - (-10)}{2} = 60 \Rightarrow \frac{60}{s} = t_{\alpha/2, 3}$
 $\Rightarrow [-10, 110] = \bar{x} \pm (2.353) s$ | $s = \frac{60}{2.353} = 25.50$ → Use to readjust with new $t_{\alpha/2, n-1}$
 $\Rightarrow$ new int = $\bar{x} \pm t_{0.025, 3} (25.50) = [-31.4, 131.4]$

• Get transporter in LEAVE? "Request" • PLAN? Cannot use to have same resource visit entity multiple times w/ diff resources each time! • Expression → Adv. Proc. Template!
 • Can use same resource in two diff. process blocks.
 • Entity is KILLED if it attempt to enter a full (capacitated) queue!
 • Sequence spreads hit → ADV Transfer • ROUTE to new station!
 • ATTRIBUTES are NOT GLOBAL! • MERGE w/ Batch

$[2W(1) - 2W(2)]/2$? → Cov. obeys signs!!
 $\text{Cov}(W(1), W(2)) = \min(s, t) = 1$
 and $\text{Var}[W(1)] = 1$, $\text{Var}[W(2)] = 2$
 and $E[W(n)] = 0$
 so $N(0, 1/4)$ ∴ $\frac{1}{2} \text{Var}[W(1) - W(2)] = 1 + 2 - (2)(1)$
 • EAR(1) → $\text{Cov}(X_0, X_1) = 1/\lambda^2$
 • $E[X_i^2] = \hat{\mu}^2 + \hat{\sigma}^2 = \bar{x}^2 + \frac{n-1}{n} S^2$
 90% → 99%? $[0, 600] = \bar{x} \pm t_{\alpha/2, b} \cdot \sqrt{\frac{S^2}{b}}$
 reps: 4
 $\bar{x} = 300$ so $t_{0.05, 3} = 3.182$
 $t_{0.025, \text{new}} = t_{0.05} \times t_{0.05} \sqrt{\frac{S^2}{b}} = (0.05) 300$ then get [,]

$Z_1^2 + Z_2^2 = \chi^2(2)$
 $= \text{Erlay}_2(1/2)$
 $\text{Exp}(1/2) + \text{Exp}(1/2)$
 $\sim \text{Erlay}_2(1/2)$
 $Z_1^2 + Z_2^2 + Z_3^2 = \chi^2(3)$
 $\Phi^{-1}(1-U) \sim N(0,1)$

Module 4: 20: ① GIGO. ② Collect Data ③ GOF tests.

INPUT TRI $\rightarrow$ (min, max, most likely) **NORM** $\rightarrow$ Height, weight, IQ's
 UNIF $\rightarrow$ not much known, min max may be.

BETA $\rightarrow$ Bounded Data **WEIBULL, GUMBEL, LOGN** $\rightarrow$ Reliability data

Can also use empirical dist!

Unbiased estimator $\rightarrow$ $Bias(T) = E[T] - \theta$
 low variance (else noise) $\rightarrow$ $E[T] = \theta$
 less bias (else overconfidence) $\rightarrow$ $E[T] = \theta$

MSE $= E[(T - \theta)^2]$
 "on avg, how far from true?"
 $MSE = Var(T) + [E(T) - \theta]^2$
 $\frac{var}{var} + \frac{bias^2}{bias^2}$

GOODNESS OF FIT Equiprob partition

$\chi^2_{stat} = \sum \frac{(O_i - E_i)^2}{E_i}$

ACCEPT if $\chi^2_{stat} < \chi^2_{k-1-s}$

Bins
 params to estimate!

$\rightarrow$ If $E_i \geq 5$, combine bins!
 $\rightarrow$ If $k-1-s$ is too big!
 $\chi^2_{stat, k-1-s} = [1 - \frac{2}{9s} + \frac{1}{9s^2}] \chi^2_{stat}$

Eq: Unif: $E_i = \frac{n}{k}$ $k-1-s$

Eq: Geom: need p but we MLE
 $\hat{p} = \frac{1}{\bar{x}}$ $\rightarrow$ just the mean!
 need $E_i = n(P(X=x)) = n(1-\hat{p})^{x-1}$
 then make table, make χ^2_{stat}

Eq: Exp: bin end $a_i = -\frac{1}{\lambda} \ln(1 - \frac{i}{k})$ if $n=100, k=5, \bar{x}=90$
 $a_i = -90 \ln(1 - \frac{i}{5})$ $(a_0, a_1] \dots (a_4, a_5]$

Eq: Weibull (γ, λ) : $\hat{\lambda}, \hat{\gamma}$ via Newton's method!
 $\hat{\lambda} = (\sum x_i^{\gamma})^{-1/\gamma}$, $\hat{\gamma} = \frac{1}{\bar{x}} [-\ln(1 - \frac{i}{k})]$

POINT ESTIMATION: MLE $= L[\theta] = \prod f(x_i; \theta)$
 MLE of θ is value that maximizes $L[\theta]$
 Use derivatives for the loge things!
 1-1 function $\Rightarrow$ Invariance property!
 Exp(λ) $\rightarrow \hat{\lambda} = \frac{1}{\bar{x}}$ $N(\mu, \sigma^2) \rightarrow \hat{\mu} = \bar{x}$
 Bern(p) $\rightarrow \hat{p} = \bar{x}$ $\Gamma(x, \lambda) \rightarrow \hat{\lambda} = \frac{\bar{x}}{\bar{x}}$

MLE may be biased but less variance!
 eg: $S^2 = \frac{n}{n-1} \hat{\sigma}^2$ **UNIF** $(0, \theta) \rightarrow \hat{\theta} = \max(x_i)$
 $L[\theta] = \begin{cases} \frac{1}{\theta^n} & 0 < x_i < \theta \\ 0 & \text{else} \end{cases}$
 if 1:1, $f(\theta)$ is MLE of $f(\theta)$
 $\Rightarrow$ never for unbiased var!! $E[S^2] = \sigma^2$
 but $\hat{\sigma} = \sqrt{\hat{\sigma}^2}$ But $E[\hat{\sigma}] \neq \sigma$

So eg: Survival func: $\hat{F}(x) = 1 - \hat{F}(x)$ CDF
 $\hat{F}(x) = e^{-\hat{\lambda}x} = e^{-\frac{x}{\bar{x}}}$
 $\Rightarrow \hat{a}_i = -\frac{1}{\hat{\lambda}} \ln(1 - \frac{i}{k}) = \bar{x} \ln(1 - \frac{i}{k})$
 $\hat{\lambda} = \frac{1}{\bar{x}}$

② Kolmogorov-Smirnov $H_0: X_1, \dots, X_n \text{ iid cdf } F(x)$
 Empirical CDF $= \hat{F}_n(x) = \frac{\# \text{ of } X_i \leq x}{n}$ Measure deviation of
 Empirical from Hypothesis!
 REJECT if Deviation $D = \max_{x \in R} |F(x) - \hat{F}_n(x)| > D_{\alpha, n}$ $\rightarrow$ get from table

Eq Unif $(0, 1)$?
 ③ Anderson-Darling
 ④ Shapiro-Wilk
 ⑥ Can use QQ plot!

Make ordered stats!

$X_{(i)}$	0.016	0.039	0.495	0.706	0.793
i/n	0.2	0.4	0.6	0.8	1
$i-1/n$	0	0.2	0.4	0.6	0.8
$\frac{i}{n} - X_{(i)}$	0.184	0.361	0.402	0.494	0.207
$X_{(i)} - \frac{i-1}{n}$	0.016	-ve	-ve	0.106	-ve

$D^+ = 0.402$
 $D^- = 0.106$
 $\max(D^+, D^-) = 0.402$
 $D_{\alpha, 5} = 0.565$ ACCEPT!

PROBLEM CHILDREN

① LITTLE Data / NO Data
 Task to Empiric!
 min, max, mode?
 Nature of obs?

② Does not fit Dist
 Min-n-match?
 Smooth the Empirical
 distro and bootstrap?

③ Non-Stationary
 Non-Homogeneous Pois?
 ARENA: "piece-wise
 constant rate fine"

④ Multivariable
 Multivari Norm?

⑤ Serially Corr (unemployment, traffic,
 ARMA, AROP, EAR(1) fatigue)

LITTLE - Good Guess
 Aves? Sums? $N(\mu, \sigma^2)$
 Bounded? Beta?
 Success/fail? Bin/Bern
 Reliability?
 Poisson? Lognormal?
 particles
 stocks

MODULE 9

INPUT Input is RV's $\rightarrow$ sim $\rightarrow$ Outputs are ESTIMATES!
 need means, variances, quantiles, probs

OUTPUT $Y_1, \dots, Y_n$ are iid but NOT independent!
 $\Rightarrow E[\bar{Y}_n] = \frac{1}{n} \sum E[Y_i] = \mu$ always unbiased!

But variance... $Var[\bar{Y}_n] = Cov(\bar{Y}_n, \bar{Y}_n) = \frac{1}{n^2} \sum_{i,j} Cov(Y_i, Y_j)$
 if $R_k = Cov(Y_i, Y_{i+k}) \forall k=0,1,2,\dots$
 $Var[\bar{Y}_n] = \frac{1}{n} [R_0 + 2 \sum_{k=1}^{n-1} (1 - \frac{k}{n}) R_k]$
 $\Rightarrow (n-1)$ not n
 Relates the sample mean to cov.
 of the process!

define $\sigma_n^2 = R_0 + 2 \sum_{k=1}^{n-1} (1 - \frac{k}{n}) R_k$
 Variance Param $\sigma^2 = \lim_{n \rightarrow \infty} \sigma_n^2 = R_0 + 2 \sum_{k=1}^{\infty} R_k = R_0 + \sum_{k=-\infty}^{\infty} R_k$
 So! if all iid, $R_k = 0 \Rightarrow \sigma^2 = \sigma_n^2 = R_0 = Var[Y_i]$
 if not, variances add up and $\sigma_n^2 >> Var(Y_i)$

Eq: 1st order Auto Regression
 $Y_i = \phi Y_{i-1} + \epsilon_i \forall i=1,2,3$
 $N(0,1) \rightarrow 1 < \phi < 1$ $N(0,1)$ $N(0, 1-\phi^2)$
 R_k here $= \phi^{|k|}$ $\forall k$
 $\Rightarrow \sigma^2 = \frac{1+\phi}{1-\phi}$ if $\phi = 0.9$, need 19 obs for one approx independent one!

SAMPLE VARIANCE
 $S_Y^2 = (\frac{1}{n-1}) \sum (Y_i - \bar{Y}_n)^2$
 if iid, unbiased for σ^2
 if not, shit.
 $E[S_Y^2] = R_0 - (\frac{2}{n-1}) \sum_{k=1}^{n-1} (1 - \frac{k}{n}) R_k$
 $\Rightarrow$ Cannot use $\frac{S_Y^2}{n}$ est, $Var[\bar{Y}_n]$

$E[S_Y^2] < Var[Y_i] \ll n Var[\bar{Y}_n]$
 if iid, $\mu \in \bar{Y} \pm t_{\frac{\alpha}{2}, n-1} \sqrt{\frac{S_Y^2}{n}}$
 if not, "true" CI will have coverage $\ll (1-\alpha)$ so be careful!

FINITE HORIZON Disc or Cont.
 Use replications! $\mu = E[\bar{Y}]$ is ok
 Variance is problem!
 Do 8 reps for 100 obs.
 Re-init w/diff random seed!
 iid: each $\odot$ is started w/same condition!
 then Grand Mean, $\bar{Z}_r = \frac{1}{r} \sum Z_i$

If (m) is large, get CLT
 Each $Z_i \sim N(0, Var(Z_i))$
 So $S_Z^2 \approx Var(Z_i) \chi^2(r-1)$
 So $\mu \in \bar{Z}_r \pm t_{\frac{\alpha}{2}, r-1} \sqrt{\frac{S_Z^2}{r}}$

Quantiles: Eg "Max wait time during rep"
 Put stuff in order: $W_{(1)} \leq W_{(2)} \leq \dots \leq W_{(n)}$
 If quantile in $\mathcal{S}_p$, estimate with floor!
 $\hat{\mathcal{S}}_p = W_{(Lp+0.5)}$
 Get CI bounds:
 $j = \lfloor rp + 0.5 - Z_{\frac{\alpha}{2}} \sqrt{rp(1-p)} \rfloor$
 $k = \lceil rp + 0.5 + Z_{\frac{\alpha}{2}} \sqrt{rp(1-p)} \rceil$
 eg $\mathcal{S}_{0.9}, r=1000 \Rightarrow \hat{\mathcal{S}}_p = W_{(900)}$
 and 95% CI = $[W_{(851)}, W_{(949)}]$

How many Reps? To reduce CI
 have $H = t_{\frac{\alpha}{2}, r-1} \sqrt{\frac{S_Z^2}{r}}$
 need $E = t_{\frac{\alpha}{2}, r-1} \sqrt{\frac{S_Z^2}{r}}$
 So $r^* = (\frac{H}{E})^2 r$
 eg: to reduce CI by factor of 10, need 100x more reps!

ARENA has an "input analyzer"

How to pick Sims! CI's Variance Red Rank & select

① CI's: Baby state can use paired CI's for

② Var. Reduction
Common PRNS!
Eg: A has time θ_A , B has θ_B
estimate $\hat{\theta}_A \rightarrow \theta_A$ $\hat{\theta}_B \rightarrow \theta_B$
ok if $\hat{\theta}_A < \hat{\theta}_B$ (eg... maybe)
BUT: $V(\hat{\theta}_A - \hat{\theta}_B) = V(\hat{\theta}_A) + V(\hat{\theta}_B)$
So lose conviction!
BUT with same PRNS,
 $V = V(\hat{\theta}_A) + V(\hat{\theta}_B) - 2\text{Cov}(\hat{\theta}_A, \hat{\theta}_B)$
you want this to drive down V!
Tighter CI's!

→ Artificial PRNS! Introduce negative correlations!
then add both and average!
 $V(\hat{\theta}_A + \hat{\theta}_B) = \frac{1}{4} [V_A + V_B + 2\text{Cov}(\hat{\theta}_A, \hat{\theta}_B)]$
eg: $V = 0.17, 0.23 \dots < \frac{V(\hat{\theta}_A)}{2}$
run sim with 0.17... → $\hat{\theta}_A$
and then (1-0.17), (1-0.23) → $\hat{\theta}_B$
Avg = $(\hat{\theta}_A + \hat{\theta}_B)/2$

③ Rank and Select: Prob of picking the best system from competitors
Indifference Zone approach: $K \sim N(\mu, \sigma^2)$
Order $\mu_{(1)} \dots \mu_{(K)}$ want to pick this!
So, say 1(CS), $\hat{p}^*$ and δ^* - smallest diff worth detecting!
Correct select: $\frac{1}{K} < \hat{p}^* < 1$ $\delta^* > 0$
if you are here, do not want to fuck up!
 $\mu_K - \mu_{K-1} \geq \delta^*$
Techniques: Bechhofer (single stop), Rinoh (two-stop), Kim and Nelson (sequen bin)

BECHHOFFER → Known common Variance NORM
Ordered means: $\mu_{(1)} = \mu_{(2)} = \dots = \mu_{(K-1)} = \mu_{(K)} - \delta$
→ SLIDAGE config: $\mu_{(K)}$ is bigger than the rest by a fixed amount!
→ LEAST FAV config: all other μ are minimized!
NORM means → Becks
BERN → Sobel and H
MULTINOM → BEM

MULTINOMIAL: K outcomes
"Best Candidate", "Next Watched", "Config that Maxes"
"k-variate Discrete Vector RV"
 $Y = (Y_1, \dots, Y_K)$ multinomial if $P(Y_1=y_1, \dots, Y_K=y_K) = \frac{n!}{y_1! \dots y_K!} p_1^{y_1} \dots p_K^{y_K}$
Eg: $R = \frac{3}{6}, G = \frac{1}{6}, B = \frac{2}{6}$
 $P(Y=(3,0,2)) = \frac{5!}{3!0!2!} (\frac{3}{6})^3 (\frac{1}{6})^0 (\frac{2}{6})^2$
for 5 x

If you don't know p_i ? for red
"Select most prob color"
- Run Sim 5x
- Use Eq for $P(i) = \pi$ to calc prob
⇒ Randomize for ties!!
The most probable outcome might be preferable to the highest Exp. Val!
P(CS) = 0.6042

STEADY-STATE SIMS Same sim. $Y_1, \dots, Y_n$ collected. Then,
 $\hat{\mu} = \bar{Y}_n = \mu$ $V(\bar{Y}_n) = \sigma^2/n$ just use $\sigma^2 = \lim_{n \rightarrow \infty} n \text{Var}(\bar{Y}_n) = \sigma^2$
Eg: MA(1) → $Y_{i+1} = \theta \epsilon_i + \epsilon_{i+1}$ $N(0,1)$
⇒ $\mu = \bar{Y}_n$ $\sigma^2 = (1+\theta)^2$
Eg: AR(1) → $Y_{i+1} = \phi Y_i + \epsilon_{i+1}$
 $\mu = \bar{Y}_n$ $\sigma^2 = \frac{1+\phi}{1-\phi}$

cannot do that? So BATCH MEANS
 $Y_1, \dots, Y_n$ $n = bm$
CLT, each batch is big ⇒ $Y_{i,m} \sim N(\mu, \sigma^2/m)$
Take 30 batches then make (m) big!

Now what? Estimate σ^2 !
Find Grand Mean $\bar{Y}_n$
 $\hat{V}_B = \left(\frac{m}{b-1}\right) \sum_{i=1}^b (\bar{Y}_{i,m} - \bar{Y})^2$
OVERLAPPING BATCH MEANS
→ Use for large n!
 $\bar{Y}_{i,m}^0 = \left(\frac{1}{m}\right) \sum_{j=1}^{i+m-1} Y_j$
 $\hat{V}_0 = \left(\frac{m}{n-m+1}\right) \sum_{i=1}^b (\bar{Y}_{i,m}^0 - \bar{Y})^2$
CRAY SHIT! $\frac{E[\hat{V}_0]}{E[\hat{V}_B]} \rightarrow 1$ but $\frac{V[\hat{V}_0]}{V[\hat{V}_B]} \rightarrow 2/3$ var is reduced!!
for large n, b batches, $d_k = d_0 = 3/2(b-1)$
 $\mu \in \bar{Y}_n \pm t_{\alpha/2} \sqrt{\frac{\hat{V}_0}{n}}$

Other way: Spectral Estimation: Use frequency domain
Regeneration: Restart sim @ points. Need lot of data!
Standardized Time Series: Stan. RVs into iid $N(\mu, \sigma^2)$
1 system! 100(1- α)% CI, μ to est. σ^2 unknown
M10 1 system: use CF 2 systems/means
 $X_1, \dots, X_m \rightarrow \sigma_x^2$ need 2
 $Y_1, \dots, Y_n \rightarrow \sigma_y^2$

POOLED CI: σ_x^2, σ_y^2 are equal but unknown
 $\mu_x - \mu_y \in \bar{X}_m - \bar{Y}_n \pm t_{\alpha/2} \sqrt{S_p^2 \left(\frac{1}{n} + \frac{1}{m}\right)}$
 $S_p^2 = \frac{(m-1)S_x^2 + (n-1)S_y^2}{(m+n-2)}$
PAIRED CI: σ_x^2, σ_y^2 unknown, unequal
 $D_i = X_i - Y_i$ iid $N(\mu_D, \sigma_D^2)$
 $\sigma_D^2 = \sigma_x^2 + \sigma_y^2 - 2\text{Cov}(X_i, Y_i)$ hope this is big!
 $\mu_D - \mu_D \in \bar{D} \pm t_{\alpha/2} \sqrt{\frac{S_D^2}{n}}$

Eg: k=3 $P(CS) \geq 0.95$, "ratio is almost 1.6"
See table, n=186
So $Y_{1,186} = 26$ $Y_{2,186} = 110$ $Y_{3,186} = 47$
winner.
Stop when col #2 cannot win → CURTAILED
Stop when there is a clear winner → SEQUENTIAL
 $N(0,1) < -1 \Rightarrow N(0,1) < -\frac{1}{\sqrt{2}}$